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ELEMENTS OF ANALYSIS

AS APPLIED TO

THE MECHANICS

OF

ENGINEERING AND MACHINERY.

TREATISE

ON

THE MECHANICS

OF

ENGINEERING AND MACHINERY,

WITH

REQUISITE ANALYTICAL INSTRUCTIONS

FOR THE USE OF

POLYTECHNIC INSTITUTES AND FOR THE REFERENCE OF ENGINEERS, ARCHITECTS, MACHINISTS, &c.

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Translated from the Fourth Revised and Enlarged German Edition.

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IN THREE VOLUMES.

VOL. I.

"THEORETICAL MECHANICS."

ILLUSTRATED WITH 902 ENGRAVINGS.

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PREFACE TO THE SECOND AMERICAN EDITION.

Dr. Weisbach's treatise on the "Principles of Mechanics" is too well known to demand any explanation of its merits at the present time. The fact that it has been translated into at least four modern languages, is sufficient proof that it supplies a want long felt by scientific men, and a decided acknowledgment of its superiority over all other works of its kind. The practical engineer here finds not only the results of continued and profound study, but is also enabled, by the clearness and precision of the author's reasonings, to review each and every step leading to those results.

The first American edition of the above work was issued in 1848, and the fact of its having been a long time out of print, while the necessity for such a work is constantly increasing, is the only consideration which has prevailed upon the translator to undertake a work requiring so much careful labor, and involving so great a responsibility. In the meantime, the original has reached its *fourth* edition, and has, by virtue of constant developments in the field of science, assumed much larger proportions. The prefaces of the author to the different editions are sufficiently explicit in regard to the nature and object of the original work, and as the translator has only sought an accurate reproduction of the same, no additional explanations are necessary.

The only essential difference between the original and the translation is the occasional substitution of the English, for the German and French, weights and measures. Yet, in all cases, the original co-efficients have been given, so that recourse may be had to them by the student, and a familiar

acquaintance with the modes of foreign calculations thus maintained. It is evident that readiness in transposing from foreign systems of computation to our own should be possessed by the practical engineer and machinist; for foreign calculations are constantly copied into our scientific publications.

Especial pains have been taken to retain the clear and accurate style of the erudite author, and if the present edition shall occupy in our country the position which the work has taken in Europe, the most sincere wish of the translator will have been attained.

AUTHOR'S PREFACE TO THE FIRST EDITION.

It is not without some hesitation that I present the First Volume of my elementary treatise on the "Mechanics of Engineering and Machinery" to the public. Although I am conscious of having composed this work with the greatest care and deliberation, I am, nevertheless, apprehensive of not having been able to satisfy the demands of all; in fact, the views and requirements of different individuals are so various as to render my task peculiarly difficult. Some will find one or another chapter too minute, whilst others will find the very same too short; some will require a more scientific treatment of certain subjects, which others would have desired to have had presented in a more popular manner. But many years of study, much experience in teaching, and manifold observation, have indicated to me the method according to which I have prepared the present work, and which I have considered the most suitable for the intended purpose.

My principle aim while preparing this work has been to attain the greatest simplicity in enunciation and proof, and to solve all important problems by means of the elementary mathematics only. When we consider the manifold knowledge to be acquired by engineers and machinists before they become capable in their departments, it is our duty as instructors, to simplify in our explanations the fundamental studies of science, by eschewing all superfluities, and by the application of the best known and most accessible auxiliaries. I have, therefore, in the present work, entirely avoided the application of the differential and integral calculus; for although the facilities for learning these methods are no longer rare, it is, nevertheless, an unquestionable fact that without constant practice the nec-

essary readiness in using them is very soon lost, and there are, therefore, many practical men, otherwise very efficient, who have forgotten how to apply them. As I am not of the same opinion as those authors who, in popular works, give the results of the more difficult problems without proofs, I have preferred to prove those which are practically important in an elementary, although sometimes prolix manner. Hence, in this work, a formula is seldom given without its derivation. Some general knowledge of certain laws of natural philosophy, and especially, a fundamental knowledge of pure elementary mathematics, is of course to be assumed in the study of this work. I have particularly endeavored to observe the just medium between *generalization* and *specialization;* for although I am not ignorant of the advantages of the former, I am, nevertheless, of the opinion that, in this, as in every elementary work, too much of it should be avoided. In practice, examples of a simple character occur more frequently than complicated ones. It is also not to be denied that, in comprehensive examples, the simple and fundamental principles are frequently out of sight, and that it is often easier to derive the compound from the simple than the simple from the compound.

This work should not be mistaken for a treatise on the construction of machines, as it is merely an introduction to the latter. Mechanics should hold the same relation to the science of the construction of machines, as descriptive geometry to the drawing of machines. After the knowledge of mechanics and descriptive geometry has been acquired, it seems most advantageous to combine the instruction on the construction and drawing of machines in one course.

There may perhaps be a doubt of the advantage of dividing this work into theoretical and practical parts; but if we consider that it is to impart instruction on all the mechanical relations in the science of architecture and machinery, the utility, or rather the necessity, of this division becomes obvious. In order to be able to judge properly of any construction, partic-

ularly of a machine, the most diverse knowledge of the laws of mechanics, as the laws of friction, strength, inertia, impact, efflux, &c., is requisite; therefore, the material for the comprehension of architecture or machinery must be gathered from nearly all the departments of mechanics. Now as it is in practice much more advantageous to study the mechanical principles of each machine in connection than to be obliged to collect them from the various departments of mechanics, the utility of the adopted division seems beyond doubt.

Having the practical application of principles constantly in view, I have sought, as far as possible, to illustrate the doctrines advanced with appropriate examples, and I can also assert that, in the great number and appropriate selection of worked examples, this work excels many others of a similar nature. I also hope that the large number of carefully executed figures will be of great service in the study of this work.

Especial attention has been devoted to the accuracy of the calculations, each example having been worked by three different persons; and it seems, therefore, hardly possible to discover any considerable errors in the same. A careful inspection of the drawings will convince the student that they have been executed with much care; in a subject of equal magnitudes, the dimensions of breadth or depth are, as a rule, made to appear only half as great as those of length and height.

Finally, it is necessary to inform the reader that he will, in this work, find much that is new, and much that is peculiar to the author. Omitting many lesser articles which occur in almost every chapter, I will call the attention of the student to the more comprehensive subjects. A universal and easy method of finding the centre of gravity of plane surfaces and plane-surfaced polyhedra may be found at §§ 107, 112, and 113; an approximate formula for the catenary at § 148; and supplements to the theory of the friction of axes at §§ 167, 168, 169, 172, and 173. The doctrine of impact has received essential

additions at §§ 277 and 278, as the impact of imperfectly elastic bodies has hitherto been too little regarded, and the case of a perfectly elastic body coming in contact with an imperfectly elastic body, not at all. The greatest number of additions, and some new laws, will be found in the department of hydraulics, the author having devoted himself to this branch for a number of years. The laws of the "Imperfect Contraction of the Fluid Vein" first observed by the author, appear here for the first time in a treatise on mechanics. The most important results of the author's experiments on the efflux of water through slides, cocks, clacks, and valves, as also through short oblique, angular, curved, and long straight, tubes, are likewise given. The chapters on flowing water, and the gauging and impulse of water have also received some additions. The theory of the reaction of effluent water, as well as that of the impulse of water according to the principle of mechanical effect, is entirely new.

Although, upon the completion of the first volume, I sometimes wish that I had treated some of the subjects in a different manner, I must, nevertheless, add that I have not yet been able to discover any essential defects. If some omissions are obvious, I must refer to the second volume, which will contain supplements, intentionally reserved for it, as has been indicated in several passages of the present volume.

My object in preparing this work has been to supply the practitioner with useful advice, the instructor with a guide in teaching, and the young engineer and machinist with a welcome auxiliary in the acquirement of a knowledge of mechanics; and it will give me great satisfaction if my purpose has been attained.

FREIBERG, *March*, 1846.

JULIUS WEISBACH.

AUTHOR'S PREFACE TO THE SECOND EDITION.

THE second edition of the first volume of the "Mechanics of Engineering and Machinery" does not, in method and arrangement, essentially differ from the first edition. It is only by the development of the work, by the changes and additions which have been made, that the present edition has been considerably extended. The enlarged appearance of this edition is especially owing to three additions. The first consists of a concise and popular introduction of the infinitesimal calculus, at the beginning of the entire work.

The object of this has been to avoid evolutions which are too complicated and elaborate by the lower calculus, and at the same time to render the student more independent in the important department of Mechanics. By the application of the rules contained in this introduction, it has been possible to take up many practically important subjects which could not be treated of, or at best, only incompletely, by elementary algebra and geometry. That those students who are not acquainted with the higher calculus may not be inconvenienced by its introduction, each paragraph in which it is applied, has been especially indicated by a parenthesis, as (§ 20).

The second addition comprises a new chapter in hydrostatics, and treats of the molecular effects of water. As the knowledge of the molecular forces is of importance in hydraulic and pneumatic observations and measurements, it has seemed to the author to be advisable to introduce the principal laws of these forces in a special chapter.

Lastly, there is an Appendix treating of oscillations and undulatory motions, inasmuch as these are of importance, on

account of the influence they exert upon the motion, strength, and durability, of machines and other constructions. It is, moreover, to the observations of oscillations that we are indebted for the most recent moduli of elasticity, which are of such very great practical importance. Magnetic force has also been considered in the Appendix, chiefly because of its great utility to the engineer in indicating his position in subterranean regions and places which afford no external view. The theory of waves, at the close of the volume, belongs wholly to the department of hydraulics; hence, no justification of its introduction in this work is necessary.

The chapter on elasticity and strength has undergone extensive changes, and received much additional matter; the department of hydraulics has also been much improved by continued experiments and corrections of the author.

May this second edition also be the recipient of the consideration and approbation with which the first was welcomed, and which has encouraged the author in the further preparation of the work.

FREIBERG, *May*, 1850.

JULIUS WEISBACH.

AUTHOR'S PREFACE TO THE FOURTH EDITION.

THE fourth edition of my "Mechanics of Engineering and Machinery," which I now submit to the public, has suffered no change either in the method or in the arrangement of the material. The somewhat rapid sale of three large editions of the work, and the appearance of two editions in the English language (in England and America), as also its translation into Swedish, Polish, and Russian, permit me to hope that this work now fully meets the exigencies of practical men, for whom it is intended. Hence, in preparing this new edition, I have confined myself to removing observed errors, and to the incorporation of new, important, empirical results, and theoretical developments. In the chapter on friction, for example, I have introduced the results of the latest experiments by Bochet, and the section which treats of elasticity and strength, has been revised with the aid of the most recent writings of Lamé, Rankine, Bressé, &c. The section on hydraulics has received manifold corrections and additions, especially, the results of recent investigations by the author. The experiments on the efflux of water under high and very high pressure, as also, those on the height of ascent of jets of water, and further, the comparative experiments on the impact of currents of air and of water, are of especial importance. The chapter on the efflux of air has been completely remodelled, the author being convinced that the general formulæ for the efflux of air under high pressure do not correctly represent the law of efflux. The formulæ obtained are very simple, as here, without diminishing the accuracy within tolerably extended limits,we have, in the known formula of heat

$$\frac{1+\delta\tau_1}{1+\delta\tau} = \left(\frac{\gamma_1}{\gamma}\right)^{0,42}$$

changed the exponent 0,42 to 0,50, and thus put

$$\frac{1+\delta\tau_1}{1+\delta\tau} = \sqrt{\frac{\gamma_1}{\gamma}} \text{ (vid. § 461).}$$

The utility of a formula depends not so much upon its accuracy for extreme limits, as upon its sufficient agreement with the results of experience, within certain limits.

Several new paragraphs have been added to the analytical laws of phoronomics and aërostatics, and in hydraulics the pressure of water flowing through tubes occupies two new paragraphs (439 and 440). In the chapter on the force and resistance of water, I have discussed the theory of the simple reaction wheel, and also its application in proving the theory of impulse and reaction of water. The more modern water and gas meters have also been treated of, as it is by the reaction of an effluent fluid that they assume a rotatory motion, the magnitude of which is easily estimated according to the foregoing theory.

Lastly, the Appendix has received a small addition in reference to Hagen's recent investigations on the subject of waves.

If it be affirmed by some that the object of this book would have been more readily attained if it had received a more scientific treatment and been based upon the higher analysis, I must here reply that the work was written especially for the private study and reference of practical men, for whom, as a rule, the requisite knowledge of the differential and integral calculus cannot be assumed.

If, finally, my "Mechanics of Engineering and Machinery" has been repeatedly used in other works of the kind, I can, since literary honor is of much greater importance to me than pecuniary profit, only rejoice; but in regard to writers who have appropriated portions of it without any acknowledgment, I can with safety appeal to the justice of public opinion.

FREIBERG, *May*, 1863. JULIUS WEISBACH.

CONTENTS.

ELEMENTS OF ANALYSIS.

Art. 1. The dependence of one magnitude y upon another x is expressed by a mathematical formula, as, for example, $y = 3\,x^2$, or $y = a\,x^m$, &c. Generally, we put $y = f(x)$, or $z = \varphi(y)$, &c., and call y a *function* of x, as also z a *function* of y. The signs f, φ, &c. merely signify generally that y depends upon x, or z upon y; they leave the dependence of these magnitudes upon each other wholly undetermined, and do not, therefore, prescribe the algebraic process by which y proceeds from x, or z from y.

A function $y = f(x)$ is an indefinite equation; there are infinitely numerous values of x and y, which correspond to the same, but if the one (x) be given, then the other (y) is determined by the function, and if the one be changed, the other, likewise, suffers a change. Hence, the indefinite magnitudes x and y are termed *variables*, or *variable magnitudes*, whilst those that are determined, or to be regarded as determined, and, therefore, prescribe the algebraic process by which y proceeds from x, are called *constants*, or *constant magnitudes*. That one of the variable magnitudes which is to be assumed arbitrarily, is termed the *independent variable*, but that one which is determined as a function of the latter by a prescribed process, is called the *dependent variable*. In $y = a\,x^m$, a and m are the constants, whilst x is the independent, and y the dependent, variable.

The dependence of one magnitude z upon two others x and y is expressed by $z = f(x, y)$. In this case z is, at the same time, function of x and y, and here, therefore, we have to consider two independent variable magnitudes.

Art. 2. Every dependence of one magnitude y upon another x, expressed by a function, or formula $y = f(x)$, may be represented by a plane curve, or curved line $A\,P\,Q$, Figs. 1 and 2; the *abscissas*

Fig. 1.

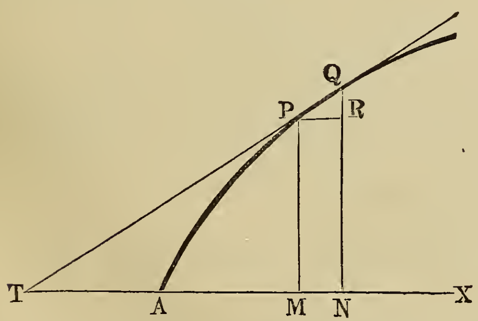
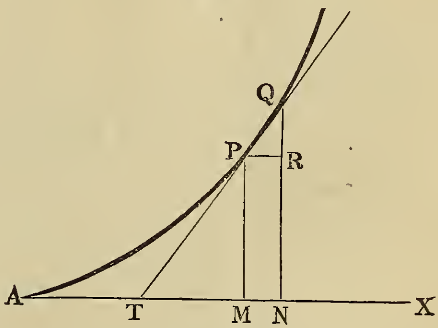


Fig. 2.

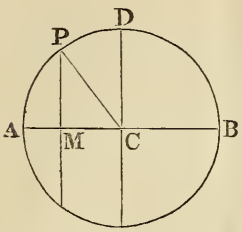


$A\,M$, $A\,N$, &c., of the curve, correspond to the different values of the independent variable x, and the *ordinates* $M\,P$, $N\,Q$, &c., to the

different values of the dependent variable y. The co-ordinates (abscissas and ordinates) of the curve, therefore, represent the two variables of the function.

The *graphic illustration* of a function, or the reduction of the same to a curve, combines many advantages: first, it affords a survey of the connection between two variable magnitudes; secondly, it takes the place of a table of two combined values of a function; and thirdly, it furnishes information in regard to the various properties and relations of functions. The circle ADB, Fig. 3, described by the radius $CA = CB = r$, which corresponds to the function $y = \sqrt{2rx - x^2}$, in which x and y represent the co-ordinates AM and MP, affords us, for example, not only a general survey of the different values which this function may assume, but also makes us acquainted with its other characteristics, since the properties of the circle have also their significance in the function. We comprehend from this, for instance, without further investigation, that y is zero, not only for $x = o$, but also for $x = 2r$; further, that y is a maximum, and indeed $= r$, if $x = r$, &c.

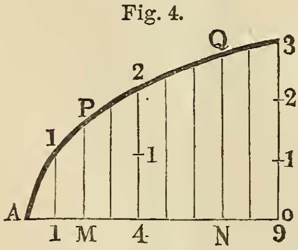
Fig. 3.



ART. 3. The *laws of natural philosophy* may, as a rule, be expressed by functions between two or more magnitudes, and are, therefore, for the most part, capable of graphic illustration.

1. For the *free descent of bodies in a vacuum*, we have, for example, for the velocity of descent y which corresponds to the height of descent x, $y = \sqrt{2gx}$; this formula agrees also with the equation $y = \sqrt{px}$ of the parabola, if the parameter (p) of the latter be put equal to twice the acceleration ($2g$) of gravity; hence, the law of descent may also be graphically illustrated by a parabola APQ, Fig. 4, with the parameter $p = 2g$. The abscissas AM, AN... of this curve are, of course, the spaces described in the descent, and the corresponding ordinates MP, NQ... the corresponding velocities.

Fig. 4.



2. If a be a certain volume of air under the pressure of one atmosphere, we have, according to the *Law of Mariotte*, the volume of the same quantity of air under a pressure of x atmospheres: $y = \frac{a}{x}$.

For $x = 1$, we have $y = a$; for $x = 2$, $y = \frac{a}{2}$; for $x = 4$, $y = \frac{a}{4}$; for $x = 10$, $y = \frac{a}{10}$; for $x = 100$, $y = \frac{a}{100}$; for $x = \infty$, $y = 0$;

thus it is clear that the volume becomes less as the tension is greater, and that, if the *Law of Mariotte* were accurate for all tensions, an infinitely small volume y would correspond to an infinitely great tension x.

Further: $x = \frac{1}{2}$ gives $y = 2\,a$; $x = \frac{1}{4}$, $y = 4\,a$;
$x = \frac{1}{10}$ „ $y = 10\,a$; $x = 0$, $y = \infty\,a$;

hence, the less the tension the greater the volume, and if the tension be infinitely small, the volume will be infinitely great.

The curve which corresponds to this law is represented in Fig. 5, (&c.); AM, AN... are the the tensions or abscissas x, MP, NQ... the corresponding volumes or ordinates y. We perceive that this curve gradually approaches the co-ordinate axes AX and AY, without ever reaching them.

3. The dependence of the *expansive force* y of *vapor* upon the temperature x may be expressed, at least within certain limits, by the formula

$$y = \left(\frac{a + x}{b}\right)^m \text{ atmospheres;}$$

and we have from experiment within certain limits, $a = 75$, $b = 175$, and $m = 6$. If, according to this, we put

$$y = \left(\frac{75 + x}{175}\right)^6,$$

Fig. 5.

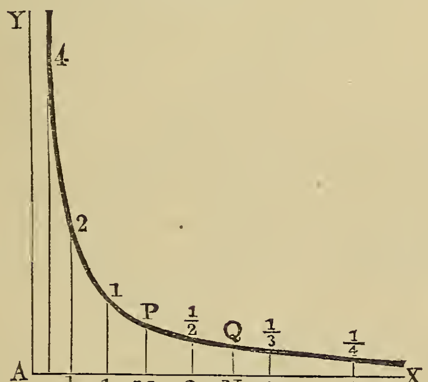


Fig. 6.

and assume perfect accuracy for this formula, we obtain:

For $x = 100^\circ$, $y = \left(\frac{175}{175}\right)^6 = 1{,}000$ atmospheres,

" $x = 50^\circ$, $y = \left(\frac{125}{175}\right)^6 = 0{,}133$ "

" $x = 0^\circ$, $y = \left(\frac{75}{105}\right)^6 = 0{,}006$ "

" $x = -75^\circ$, $y = \left(\frac{0}{175}\right)^6 = 0{,}000$ "

$$\text{further, for } x = 120^\circ,\quad y = \left(\frac{195}{175}\right)^6 = 1{,}914 \text{ atmospheres,}$$

$$\text{" } x = 150^\circ,\quad y = \left(\frac{225}{175}\right)^6 = 4{,}517 \quad \text{"}$$

$$\text{" } x = 200^\circ,\quad y = \left(\frac{275}{175}\right)^6 = 15{,}058 \quad \text{"}$$

The corresponding curve is represented by PQ, Fig. 6; it passes through the axis of abcissas at a distance $AO = -75$ from the origin A of the co-ordinates, and through the axis of ordinates, at a distance $AS = 0{,}006$ from this same point; further, the ordinate $MP < 1$ corresponds to the abscissa $AM < 100$, and the ordinate $NQ > 1$, to the abscissa $AN > 100$; and it may also be observed, not only that y increases to infinity with x, but also, that the curve ascends more and more perpendicularly the greater x becomes.

ART. 4. A function $z = f(x, y)$ with two *independent variables* may be illustrated by a *curved surface* BCD, Fig. 7, in which these variables x and y are represented by the abscissas AM and AN on the axes AX and AY, and the dependent variable z, by the ordinate OP of a point P in the surface ABC. If, with a determined value of x, we give different values to y, we obtain in z the ordinates of the points of a curve EPF running parallel with the plane of co-ordinates YZ; but if, with a determined value of y, we assume different values for x, there results in z the ordinates of the points of a curve GPH running parallel with the plane of co-ordinates XZ. Therefore, the entire curved surface BCD may be regarded as a continuous combination of curves running parallel with the planes of the co-ordinates.

The *Law of Mariotte, Gay*, and *Lussac, viz.* $z = \frac{a(1 + \delta y)}{x}$, according to which the volume z of a quantity of air may be calculated by the pressure x and temperature y of the same, may be graphically illustrated by the curved surface $CKPH$, Fig. 8. AM is the pressure x, $ANMO$ the temperature y, and OP the corresponding

Fig. 7.

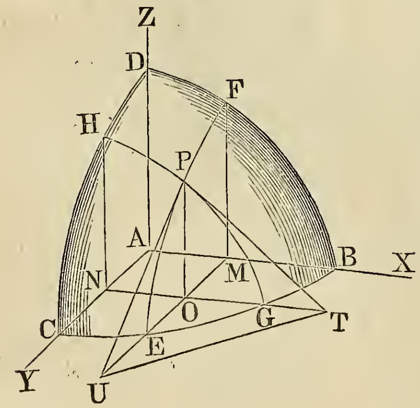
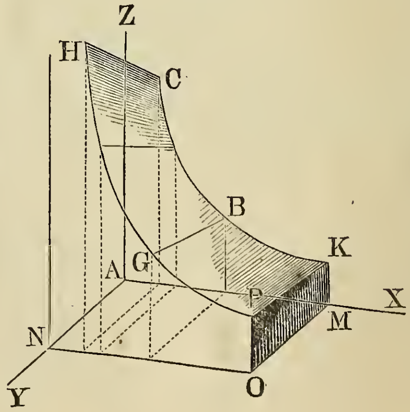


Fig. 8.



volume z; further, the co-ordinates of the curve PGH indicate the volumes for one and the same temperature $AN = y$, and the co-ordinates of the straight line KP, the volumes for one and the same pressure $AM = x$.

Art. 5. If the independent variable of a function or abscissa $AM = x$, Figs. 9 and 10, of the corresponding curve be allowed to increase by an infinitesimal magnitude MN, which we shall in future designate by ∂x, the corresponding dependent variable, or ordinate $MP = y$ passes into $NQ = y_1$, and increases by the infinitesimal value $RQ = NQ - MP$, which is to be designated by ∂y.

Both increments ∂x and ∂y of x and y are called *differentials*, or *elements* of the variables or co-ordinates x and y, and it is now our chief task to find, for the most frequently recurring functions, the differentials, or rather, the relations between the associated elements of their variables x and y. If, in the function $y = f(x)$, where x designates the abscissa AM, and y, the ordinate MP, we replace x by $x + \partial x = AM + MN = AN$, we obtain, instead of y, $y + \partial y = MP + RQ = NQ$; therefore:

$$y + \partial y = f(x + \partial x);$$

and if, from this we abstract the first value of y, there will remain the *differential of the variable* y; i. e.:

$$\partial y = \partial f(x) = f(x + \partial x) - f(x).$$

Fig. 9.

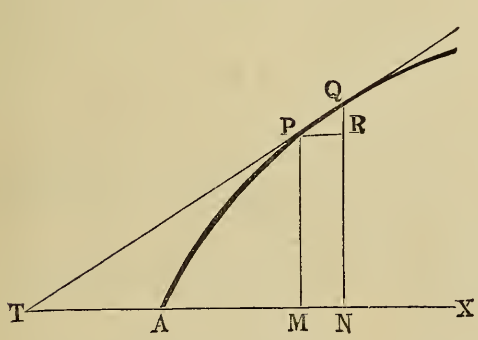
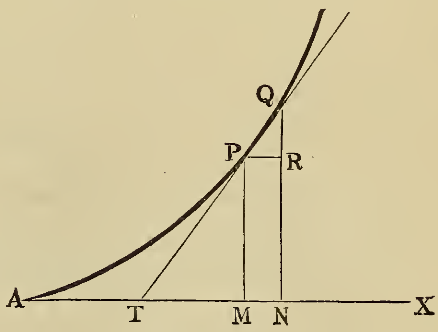


Fig. 10.



This is the most *general rule* for the determination of the differential of a function, from which, by application to different functions, other rules more or less general may be deduced.

If we have, for instance, $y = x^2$, then

$$\partial y = (x + \partial x)^2 - x^2,$$

or, since we have to put

$$(x + \partial x)^2 = x^2 + 2x\partial x + \partial x^2,$$

there will result

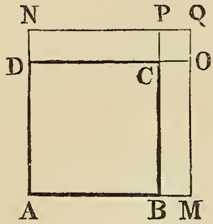
$$\partial y = 2x\partial x + \partial x^2 = (2x + \partial x)\,\partial x;$$

or more simply, since ∂x, as infinitesimal, vanishes when compared with $2x$:

$$\partial y = \partial (x)^2 = 2x\partial x.$$

$y = x^2$ corresponds to the area of a square $ABCD$, Fig. 11, whose side $AB = AD$ is $= x$; and we may infer from the figure, that, by the augmentation of the sides by $BM = DN = \partial x$, the square is enlarged by two rectangles BO and $DP = 2x\partial x$, and a square $OP = (\partial x)^2$; that, therefore, with an infinitesimal increment (∂x) of x, the square $y = x^2$ increases by the element $2x\partial x$.

Fig. 11.



ART. 6. The straight line TPQ, Figs. 9 and 10, which passes through two infinitely approximate points P, Q, of a curve, is called a *tangent*, or *line of contact* of this curve, and determines the direction of the same between these points. The direction of the tangent is indicated by the angle $PTM = \alpha$ under which the axis AX of abscissas is intersected by this line. If the curve be *concave*, as APQ, Fig. 9, the tangent will lie outside the curve and axis of abscissas; but if the curve be *convex*, as APQ, Fig. 10, it will lie between the curve and axis of abscissas.

In the infinitesimal right-angled triangle PQR, Figs. 9 and 10, having the catheti $PR = \partial x$ and $RQ = \partial y$, the angle QPR is equal to the *tangential angle* $PTM = \alpha$, and, since we have

$$tang.\ QPR = \frac{QR}{PR},$$

we have also:

$$tang.\ \alpha = \frac{\partial y}{\partial x};$$

therefore, the *quotient of the two differentials ∂y and ∂x indicates the trigonometric tangent of the tangential angle.*

For the parabola, for instance, whose equation is $y^2 = px$, we have, if we put $y^2 = px = z$,

$$\partial z = (y + \partial y)^2 - y^2 = y^2 + 2y\partial y + \partial y^2 - y^2 = 2y\partial y + \partial y^2,$$

or, since ∂y^2 vanishes when compared with $2y\partial y$:

$$\partial z = 2y\partial y,$$

and likewise:

$$\partial z = p(x + \partial x) - p\partial x.$$

There results, accordingly, $2y\partial y = p\partial x$, and hence, for the tangential angle of the parabola:

$$tang.\ \alpha = \frac{\partial y}{\partial x} = \frac{p}{2y} = \frac{y^2}{2xy} = \frac{x}{2y}.$$

Generally, we term the determined portion PT of the line of contact between the point of contact P and the point of intersection T

Fig. 12.

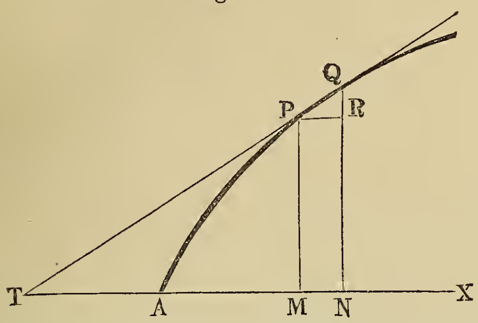
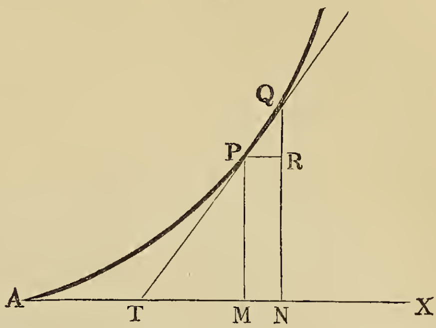


Fig. 13.



with the axis of abscissas, the *tangent*, and its projection TM in the axis of abscissas, the *subtangent*, and we thus have

$$subtang. = PM\ cotang.\ PTM$$
$$= y\ cotang.\ \alpha = y\,\frac{\partial x}{\partial y};$$

for example, for the parabola:

$$subtang. = y \,.\, \frac{2x}{y} = 2\,x.$$

Here, therefore, the subtangent is equal to twice the abscissa, and the position of the tangent for every point P of the parabola is, accordingly, easily indicated.

For a *curved surface* BCD Fig. 7, the angles of inclination α and β of the tangents PT and PU at a point P are determined by the formulæ:

$$tang.\ \alpha = \frac{\partial z}{\partial x},\ \textit{and}\ tang.\ \beta = \frac{\partial z}{\partial y}.$$

The plane PTU laid through PT and PU is a tangential plane of the curved surface.

ART. 7. For a function $y = a + mf(x)$, we have:

$$\partial y = [a + mf(x + \partial x)] - [a + mf(x)]$$
$$= a - a + mf(x + \partial x) - mf(x)$$
$$= m\,[f(x + \partial x) - f(x)];\ \text{i. e.:}$$

$$\text{I.} \quad . \ . \ . \ . \ . \ . \quad \partial\,[a + mf(x)] = m\partial f(x);$$

e. g.:

$$\partial\,(5 + 3\,x^2) = 3\,[(x + \partial x)^2 - x^2] = 3 \,.\, 2\,x\partial x = 6\,x\partial x.$$

There is likewise:

$$\partial\,(4 - \tfrac{1}{2}\,x^3) = -\tfrac{1}{2}\,\partial\,(x)^3 = -\tfrac{1}{2}\,[(x + \partial x)^3 - x^3]$$
$$= -\tfrac{1}{2}\,(x^3 + 3\,x^2\,\partial x + 3\,x\partial x^2 + \partial x^3 - x^3)$$
$$= -\tfrac{1}{2} \,.\, 3\,x^2\partial x = -\tfrac{3}{2}\,x^2\partial x.$$

We may, accordingly, adduce the following important rule: *The constant members* (a, 5) *of a function vanish by differentiation, and the constant factors* (m, 3) *remain thereby unchanged.*

The accuracy of this rule may also be graphically illustrated. For the curve APQ, Fig. 14, whose co-ordinates are first, $AM = x$ and

Fig. 14.

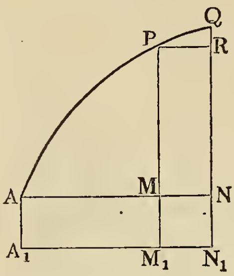
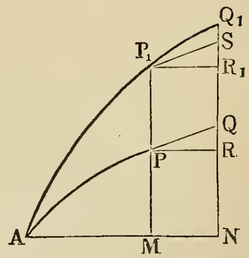


Fig. 15.



$MP = y = f(x)$, and afterwards, $A_1 M_1 = x$ and $M_1 P = a + y = a + f(x)$, we have $PR = \partial x$ and $RQ = \partial y = \partial f(x)$, and also $= \partial (a+y) = \partial[a + f(x)]$; and for the curves $AP_1 Q_1$ and APQ, Fig. 15, whose associated ordinates MP_1 and MP, NQ_1 and NQ have a certain relation to each other, the relation between the differentials

$R_1 Q_1 = NQ_1 - MP_1$ and $RQ = NQ - MP$ is always the same; for if we put $MP_1 = m \cdot MP$, and $NQ_1 = m \cdot NQ$, there follows

$$R_1 Q_1 = NQ_1 - MP_1 = m(NQ - MP) = m \cdot QR;$$

i. e.:

$$\partial[mf(x)] = m\partial f(x).$$

If we have, further, $y = u + v$, therefore, the *sum of two variables* u and v, there is also

$$\partial y = u + \partial u + v + \partial v - (u + v); \text{ i. e., from Art. 5:}$$

$$\text{II.} \quad \ldots\ldots \quad \partial(u+v) = \partial u + \partial v; \text{ likewise:}$$
$$\partial[f(x) + \varphi(x)] = \partial f(x) + \partial \varphi(x).$$

Thus, *the differential of the sum of several functions is equal to the sum of the differentials of the single functions;* e. g.:
$\partial(2x + 3x^2 - \frac{1}{2}x^3) = 2\partial x + 6x\partial x - \frac{3}{2}x^2 \partial x = (2 + 6x - \frac{3}{2}x^2)\partial x$.

The accuracy of this rule is also verified in the curve APQ, Fig. 15. If we have $MP = f(x)$ and $PP_1 = \varphi(x)$, there is also

$$MP_1 = y = f(x) + \varphi(x), \text{ and}$$

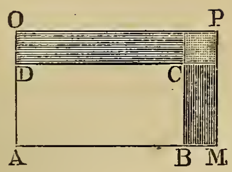
$$\partial y = R_1 Q_1 = R_1 S + SQ_1 = RQ + SQ_1 = \partial f(x) + \partial \varphi(x);$$

as we may put $P_1 S$ parallel to PQ, and hence $R_1 S = RQ$ and $QS = PP_1$.

Art. 8. If we have $y = uv$, therefore, the product *of two variables*, as the area of a rectangle $ABCD$, Fig. 16, with the variable sides $AB = u$ and $BC = v$, there results

$$\partial y = (u + \partial u)(v + \partial v) - uv = uv + u\partial v + v\partial u + \partial u \partial v - uv$$
$$= u\partial v + v\partial u + \partial u\partial v = u\partial v + (v + \partial v)u.$$

Fig. 16.



But in $v + \partial v$, ∂v is infinitesimal in comparison with v; hence we may put

$$v + \partial v = v, \text{ and } (v + \partial v)\partial u = v\partial u,$$

as also,

$$u\partial v + (v + \partial v)\partial u = u\partial v + v\partial u;$$

so that there follows:

$$\text{III.} \quad . \quad . \quad . \quad \partial(uv) = u\partial v + v\partial u,$$

and

$$\partial[f(x) \,.\, \varphi(x)] = f(x)\partial\varphi(x) + \varphi(x)\partial f(x).$$

Therefore, *the differential of the product of two variables is equal the sum of the products of the one, and the differential of the other, variable.*

If the sides of the rectangle $ABCD$, Fig. 16, increase by $BM = \partial u$ and $DO = \partial v$, the area $y = AB \,.\, AD = uv$ of the same will increase by the rectangles $CO = u\partial v$, $CM = v\partial u$, and $CP = \partial u \partial v$, the last of which, being infinitely small in comparison with the others, vanishes; hence, the differential of this area is only to be put equal to the sum $u\partial v + v\partial u$ of the areas of the two rectangles CO and CM.

According to this rule, we have, for instance, for $y = x(3x^2 + 1)$:

$$\partial y = x\partial (3x^2 + 1) + (3x^2 + 1)\partial x = 3x\partial(x^2) + (3x^2 + 1)\partial x$$
$$= 3x \,.\, 2x\partial x + 3x^2\partial x + \partial x = (9x^2 + 1)\partial x.$$

There is further, if w designate a third variable factor:

$$\partial(uvw) = u\partial(vw) + vw\partial u; \text{ or,}$$

since we have $\partial(vw) = v\partial w + w\partial v$:

$$\partial(uvw) = uv\partial w + uw\partial v + vw\partial u; \text{ likewise,}$$
$$\partial(uvwz) = uvw\partial z + uvz\partial w + uwz\partial v + vwz\partial u.$$

If we have $u = v = w = z$, there follows $\partial(u^4) = 4u^3\partial u$, as also generally:

$$\text{IV.} \quad . \quad . \quad . \quad . \quad \partial(x^m) = mx^{m-1}\partial x,$$

if the exponent m be a *positive whole number.*

For example: $\partial(x^7) = 7x^6\partial x$, and $\partial(\frac{3}{4}x^8) = 6x^7\partial x$.

If, in $y = x^{-m}$, m be again a positive whole number, we have also

$$yx^m = 1, \text{ and } \partial(yx^m) = 0; \text{ i. e.:}$$
$$y\partial(x^m) + x^m\partial y = 0, \text{ and hence,}$$

$$\partial y = -\frac{y\partial(x^m)}{x^m} = -\frac{x^{-m} \,.\, mx^{m-1}\partial x}{x^m} = -mx^{-m-1}\partial x;$$

or, if we put $-m = n$:

$$\partial(x^n) = nx^{n-1}\partial x.$$

Therefore, rule IV. applies also to powers with *negative whole numbers* for *exponents.* For example:

$$\partial(x^{-3}) = -3x^{-4}\partial x = -\frac{3\partial x}{x^4};$$

likewise:

$$\partial\,(3\,x^2+1)^{-2}=-\,2\,(3\,x^2+1)^{-3}\,\partial\,(3\,x^2)=-\,\frac{12x\,\partial x}{(3\,x+1)^3}.$$

If in $y=x^{\frac{m}{n}}$, $\frac{m}{n}$ be a fraction whose denominator n and numerator m are whole numbers, we have also $y^n=x^m$, and $\partial\,(y^n)=\partial\,(x^m)$; i. e.: $n y^{n-1}\,\partial y=m x^{m-1}\,\partial x$; hence,

$$\partial y=\frac{m}{n}\,\frac{x^{m-1}\,\partial x}{y^{n-1}}=\frac{m}{n}\,\frac{x^{m-1}\,\partial x}{x^{m-\frac{m}{n}}}=\frac{m}{n}\,x^{\frac{m}{n}-1}\,\partial x.$$

If we put $\frac{m}{n}=p$, there follows

$\partial y=\partial\,(x^p)=p\,x^{p-1}\,\partial x$, which likewise corresponds to rule IV., now generally regarded as accurate.

We have also $\partial\,(u^p)=p\,u^{p-1}\,\partial u$, if u designate some dependent function of x.

From this we have, for instance:

$$\partial\,(\sqrt{x^3})=\partial\,(x^{3/2})=\tfrac{3}{2}x^{1/2}\,\partial x=\tfrac{3}{2}\sqrt{x}\,\partial x,$$

$$\partial\,\sqrt{2rx-x^2}=\partial\,\sqrt{u}=\partial\,(u^{1/2})=\tfrac{1}{2}u^{-1/2}\,\partial u$$

$$=\tfrac{1}{2}\,\frac{\partial\,(2rx-x^2)}{u^{1/2}}=\frac{2r\partial x-2x\partial x}{2\sqrt{u}}=\frac{(r-x)\,\partial x}{\sqrt{2rx-x^2}}.$$

To find the *differential of a quotient* $y=\frac{u}{v}$, let us put $u=vy$, from which $\partial u=v\partial y+y\partial v$; consequently, there follows

$$\partial y=\frac{\partial u-y\partial v}{v}=\frac{\partial u-\frac{u}{v}\,\partial v}{v};\text{ i. e.:}$$

$$\text{V.}\quad \partial\left(\frac{u}{v}\right)=\frac{v\partial u-u\partial v}{v^2}.$$

We have, therefore, for example:

$$\partial\left(\frac{x^2-1}{x+2}\right)=\frac{(x+2)\,\partial\,(x^2-1)-(x^2-1)\,\partial\,(x+2)}{(x+2)^2}$$

$$=\frac{(x+2)\,.\,2x\partial x-(x^2-1)\,.\,\partial x}{(x+2)^2}=\left(\frac{x^2+4x+1}{(x+2)^2}\right)\partial x;$$

and further,

$$\partial\left(\frac{a}{v}\right)=-\,\frac{a\,\partial v}{v^2};\text{ e. g.: }\partial\left(\frac{4}{x^2}\right)=-\,\frac{4\,\partial(x^2)}{x^4}=-\,\frac{8\,\partial x}{x^3}.$$

ART. 9. The function $y=x^n$ is the most important of the entire analysis, as it occurs in almost every investigation. If, to the exponent n we ascribe every possible value, positive and negative, entire and fractional, &c., it will also produce the most diverse curves, as may be seen in Fig. 17. Here, A is zero, or the origin of the coordinate axes $X\overline{X}$ and $Y\overline{Y}$.

If, at either side of the co-ordinate axes, at the distances $x = \pm 1$ and $y = \pm 1$ from A, we draw the lines $X_1 \overline{X}_1$, $X_2 \overline{X}_2$, $Y_1 \overline{Y}_1$, and $Y_2 \overline{Y}_2$ parallel to these axes, and connect their points of intersection P_1, P_2, P_3, and P_4 by the transversals $Z\overline{Z}$, $Z_1 \overline{Z}_1$, we obtain a diagram with which the curves corresponding to the equation $y = x^n$ are more or less connected. Moreover, we have for every point of the axis of abscissas $X\overline{X}$, $y = 0$, as also for every point of the axis of ordinates $Y\overline{Y}$, $x = 0$; further, for the points in the axes $X_1 \overline{X}_1$ and $X_2 \overline{X}_2$, $y = \pm 1$, and for those in the axes $Y_1 \overline{Y}_1$ and $Y_2 \overline{Y}_2$, $x = \pm 1$.

If, in the equation $y = x^n$, we put $x = 1$, we always obtain $y = 1$, no matter what kind of a number the exponent n may be, and only for certain infrequent values of n, can $y = -1$. Consequently, all the curves proper to the equation $y = x^n$ pass through the point P_1 having the co-ordinates $AM = 1$ and $AN = 1$.

If we assume $n = 1$, and thus make $y = x$, we obtain the straight line $ZA\overline{Z}$, deviating equally from the two axes $X\overline{X}$ and $Y\overline{Y}$, which ascends on the one side of A at an angle of $45^0 \left(\frac{\pi}{4}\right)$, and descends at the same angle on the other side. On the contrary, for $y = -x$, we obtain the straight line $Z_1 A \overline{Z}_1$, descending at an angle of 45^0 on the one side of A, and ascending at the same angle on the other.

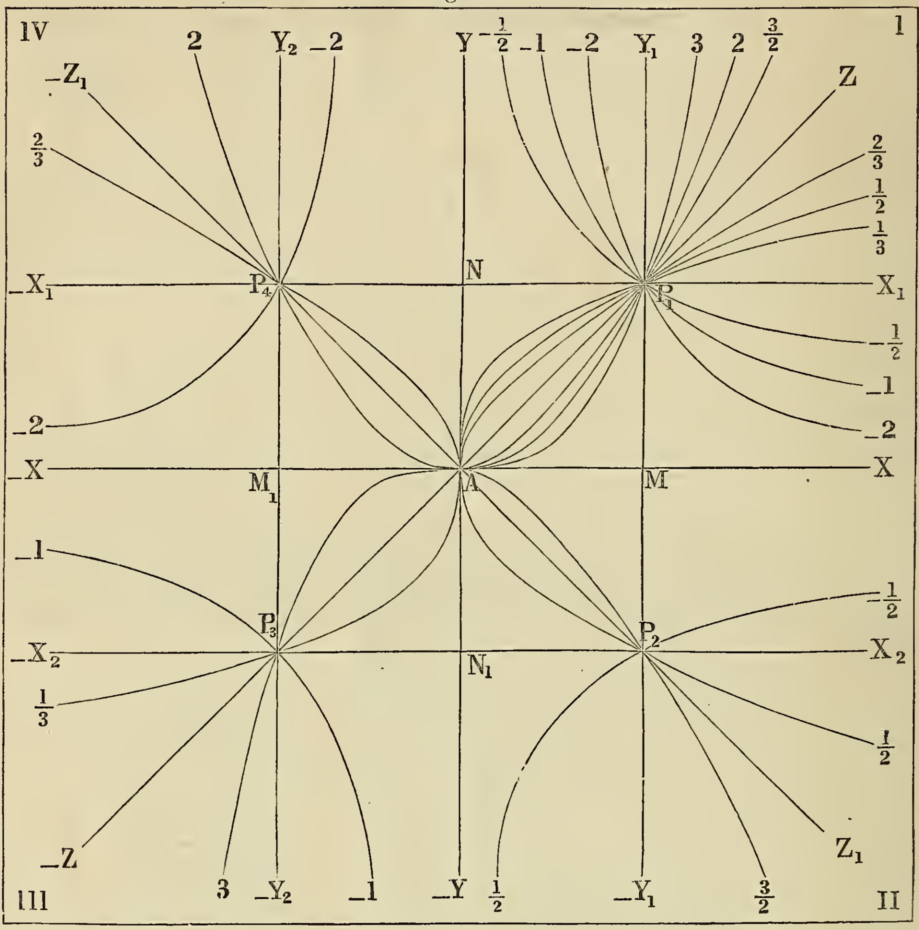
If, on the other hand, we have $n > 1$, $y = x^n$ must be less than x for $x < 1$, and greater than x for $x > 1$; and if we have $n < 1$, $y = x^n$ must be greater than x for $x < 1$, and less than x for $x > 1$.

To the first case ($n > 1$) correspond *convex curves*, which, at the beginning, run below the straight line $ZA\overline{Z}$, but which, from P_1, run above the same, whilst to the second case ($n < 1$) correspond *concave curves* in which the reverse occurs.

If, in the first case, the exponent n be assumed as becoming less and less and finally vanishing, or approximating zero, the ordinates will approximate more and more to the constant value $y = x^0 = 1$, and the corresponding curves above AX, to the broken line ANP_1X_1; but if, in the second case, the exponent n become greater and greater, the ordinates will gradually approximate to the limiting value $y = x^\infty = x^{\frac{1}{0}} = \infty$, whilst the abscissas will gradually approximate to the limit $x = y^0 = 1$; and hence, the corresponding curves will approach nearer and nearer to the broken line $AMP_1 Y_1$.

If we assume $n = -1$, thus putting $y = x^{-1} = \frac{1}{x}$, we shall have for $x = 0$, $y = \infty$, and for $x = \infty$, $y = 0$; and we have therefore to consider a curve $\overline{1}\, P_1\, \overline{1}$ (treated of in Art. 3, and illustrated in Fig. 5,) which always approaches the axis of ordinates on the one side, and the axis of abscissas on the other, but without ever reaching either the one or the other.

Fig. 17.



If the exponent $(-n)$ of the function $y = x^{-n} = \frac{1}{x^n}$ be a proper fraction, we shall have for $x < 1$, $y < \frac{1}{x}$, and for $x > 1$, $y > \frac{1}{x}$; but if this exponent be greater than unity, we shall have for $x < 1$, $y > \frac{1}{x}$, and for $x > 1$, $y < \frac{1}{x}$. Therefore, the curves corresponding to the function $y = x^{-n}$ run, at the beginning, below or above, and afterwards, from the point P, above or below the curve $y = x^{-1} = \frac{1}{x}$, according as n is less or greater than unity. Whilst generally the curves corresponding to the positive values of n run first below,

and from P_1, above, the straight line $X_1 \overline{X}_1$, those proceeding from negative exponents $(-n)$, pass first above, and from P_1, below, the line $X_1 \overline{X}_1$. In the former curves, we have for $x = 0$, also $y = 0$, and for $x = \infty$, also $y = \infty$; whilst in the latter, we have for $x = 0$, $y = \infty$, and for $x = \infty$, $y = 0$. If the former depart more and more from the co-ordinate axes $X\overline{X}$ and $Y\overline{Y}$, the farther we follow them from the origin A, the latter approach more and more the axis $X\overline{X}$ on the one side, and the axis $Y\overline{Y}$ on the other, but without ever reaching them.

Moreover, the last systems of curves approach nearer and nearer the broken line YNP_1X_1, or the broken line Y_1P_1MX, according as the exponent approximates the limit $n = 0$ or $n = \infty$.

If, in $y = x^{\pm m}$, m be an *odd whole number* (1, 3, 5, 7 . . .), y has the same sign as x; also, positive values of y correspond to positive values of x, and negative values of y, to negative values of x. If, on the other hand, m be an *even whole number* (2, 4, 6 . . .), y will be positive for both positive and negative x. The curves in the first case (as $3\ P_1\ AP_3\ 3$, or $\overline{1}\ P_1\ \overline{1}$, $\overline{1}\ P_3\ \overline{1}$) run, consequently, on the one side of the axis of ordinates, above, and on the other, below, the axis of abscissas $XA\overline{X}$; in the second case, however, (as $2\ P_1\ A\ P_4\ 2$, or $\overline{2}\ P_1\ \overline{2}$, $\overline{2}\ P_4\ \overline{2}$) the curves run only above the axis of abscissas, and occupy, therefore, only the first and fourth squares. The former, for $m = \pm\infty$, correspond to the limiting lines $Y_1\,MAM_1\overline{Y}_2$ and $X\,M\,Y_1$, $\overline{X}M_1\overline{Y}_2$, whilst the latter correspond to the limiting lines $Y_1\,MA\,M_1\,Y_2$ and $X\,M\,Y_1$, $\overline{X}\,M_1\,Y_2$.

If, in $y = x^{\pm\frac{1}{n}}$, n be an *odd whole number*, y has the same sign as x; and if n be an *even whole number*, every positive x will give two equal values for y, one positive, and one negative, whilst for every negative x, y is imaginary or impossible. The curves (as $\frac{1}{3}\ P_1\ A\ P_3\ \frac{1}{3}$) which correspond to the first case, are, therefore, only to be found in the first and third squares, and those for the second case (as $\frac{1}{2}\ P_1\ AP_2\ \frac{1}{2}$), only in the first and second; the former have, for $m = \infty$, the lines of limitation $X_1\,NA\,N_1\overline{X}_2$ and $X_1\,N\,Y$, $\overline{X}_2\,N_1\,\overline{Y}$; the latter, the lines $X_1\,NA\,N_1\,X_2$ and $X_1\,N\,Y$, $X_2\,N_1\,\overline{Y}$.

As $y = x^{\pm\frac{1}{n}}$ requires $x = y^{\pm n}$, it follows that the last system of curves $\left(y = x^{\pm\frac{1}{n}}\right)$ does not deviate from the foregoing $\left(y = x^{\pm m}\right)$, except in its position in regard to the intersection of the axes, and

that, by turning the curves, those of the one system may be brought to coincide with those of the other.

Since we have $y = x^{\frac{m}{n}} = \left(x^{\frac{1}{n}}\right)^{m} = (x^{m})^{\frac{1}{n}}$, the general course of the corresponding curve may always be given from the foregoing. For example, the curve for

$$y = x^{\frac{2}{3}} = (x^{\frac{1}{3}})^{2} = (\sqrt[3]{x})^{2},$$

has positive ordinates both for positive and for negative x. On the other hand, the curve for

$$y = x^{\frac{3}{2}} = (x^{\frac{1}{2}})^{3} = (\sqrt{x})^{3},$$

has positive ordinates for positive x only, and the two are, indeed, opposite. Further, in the curve for

$$y = x^{\frac{3}{5}} = (\sqrt[5]{x})^{3},$$

y has always the same sign as x, as neither the fifth root nor the cube changes the sign of the base.

Lastly, the curves corresponding to the equation $y = - x^{\frac{m}{n}}$, differ from those of the equation $y = x^{\frac{m}{n}}$, only in their opposite positions in reference to the axis of abscissas $X\overline{X}$, and constitute the symmetrical halves of a whole.

Art. 10. From the important formula $\partial\,(x^{n}) = n\,x^{n-1}\,\partial\,x$ there follows also the formula for the *tangential angle* of the corresponding curves, illustrated in Fig. 18; we have, namely:

$$tang.\ a = \frac{\partial y}{\partial x} = n\,x^{n-1},$$

and hence, the subtangent of these curves:

$$= y\,\frac{\partial x}{\partial y} = \frac{x^{n}}{n\,x^{n-1}} = \frac{x}{n}.$$

There is, accordingly, for *Neil's* parabola, whose equation is $a\,y^{2} = x^{3}$, or $y = \sqrt{\frac{x^{3}}{a}}$:

$$tang.\ a = \frac{1}{\sqrt{a}}\,\frac{\partial\,(x^{\frac{3}{2}})}{\partial\,x} = \frac{1}{\sqrt{a}}.\ \tfrac{3}{2}\,x^{\frac{1}{2}} = \tfrac{3}{2}\sqrt{\frac{x}{a}},$$

and the subtangent $= \frac{2}{3}\,x$.

Further, we have already for the above curve $y = \frac{a^{2}}{x} = a^{2}\,x^{-1}$:

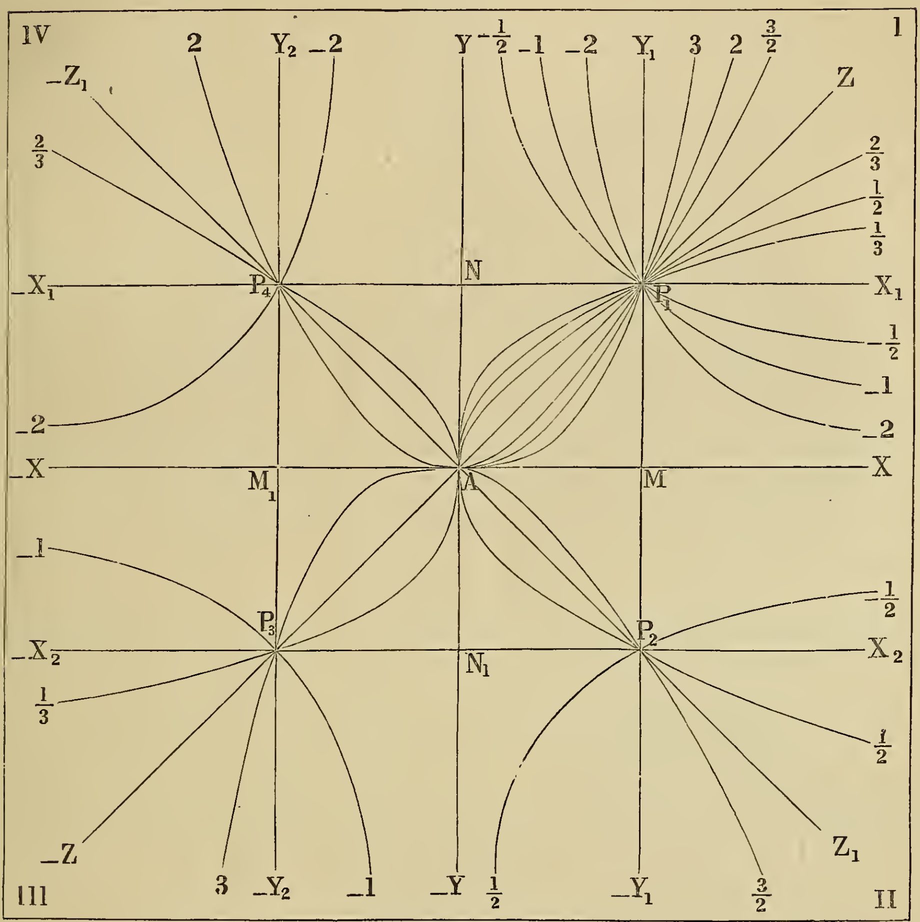
$$tang.\ a = a^{2}\,\frac{\partial\,(x^{-1})}{\partial\,x} = -\,\frac{a^{2}}{x^{2}} = -\left(\frac{a}{x}\right)^{2},$$

and the subtangent $= \frac{x}{-1} = -\,x$. (Comp. Fig. 5.)

Consequently, we shall have

$$\text{for } x = 0,\ tang.\ a = -\infty, \text{ therefore, } a = 90^0;$$
$$\text{further, for } x = a,\ tang.\ a = -1, \text{ therefore, } a = 135^0;$$
$$\text{and for } x = \infty,\ tang.\ a = 0, \text{ therefore, } a = 0^0; \&c.$$

Fig. 18.



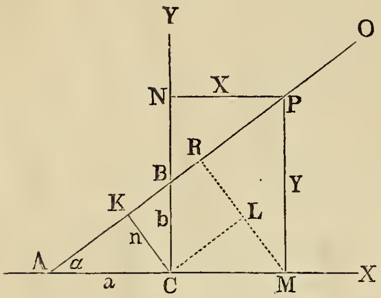
ART. 11. If a *straight line* AO, Fig. 19, intersects the axis of abscissas under the angle $OAX = a$, and is distant from the origin C of the co-ordinates by $CK = n$, the equation between the co-ordinates $CM = NP = x$ and $CN = MP = y$ of a point P in the line, will be, since we have $n = MR - ML$ and $MR = y \cos. a$, as also $ML = x \sin. a$,

$$y \cos a - x \sin. a = n.$$

For $x = 0$, y assumes the value $CB = b = \dfrac{n}{\cos. a}$; hence, we have also $n = b \cdot \cos. a$, and $y \cos. a - x \sin. a = b \cos. a$, or $y = b + x\ tang.\ a$.

The lines CA and CB by which the points of intersection A and B of the straight line with the co-ordinate axes CX and CY, are

Fig. 19.



distant from the origin C, are generally termed the *parameters* of the straight line, and are designated by the letters a and b. According to the figure, we have $CA = -a$, hence:

$$tang.\ \alpha = \frac{CB}{CA} = -\frac{b}{a},$$

and consequently, the equation of the straight line: $y = b - \frac{b}{a} x$, or:

$$\frac{x}{a} + \frac{y}{b} = 1;\ \text{(vid. Ingenieur. page 164).}$$

When a curve approaches nearer and nearer, *ad infinitum*, to a straight line which is distant by a finite magnitude from the origin of the co-ordinates, without ever reaching the same, this straight line is called the *asymptote of the curve.*

The asymptote may be regarded as the tangent or line of contact for an infinitely distant point of the curve. Its angle of inclination α to the axis of abscissas is, therefore, determined by

$$tang.\ \alpha = \frac{\partial y}{\partial x},$$

and its distance n from the zero of the co-ordinates, by the equation

$$n = y\ cos.\ \alpha - x\ sin.\ \alpha = (y - x\ tang.\ \alpha)\ cos.\ \alpha$$

$$= \frac{y - x\ tang.\ \alpha}{\sqrt{1 + (tang.\ \alpha)^2}} = \left(y - x\frac{\partial y}{\partial x}\right) : \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2},$$

as also by $n = (y\ cotg.\ \alpha - x)\ sin.\ \alpha = \dfrac{y\ cotg.\ \alpha - x}{\sqrt{1 + (cotg.\ \alpha)^2}}$

$$= \left(y\frac{\partial x}{\partial y} - x\right) : \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2},$$

if we put x and $y = \infty$.

In order that a tangent for an infinitely distant point of contact may be an asymptote, it is necessary, that, for x or $y = \infty$, $y - x\ tang.\ \alpha$, or $y\ cotg.\ \alpha - x$, be not infinitely great.

For a curve of the equation $y = x^{-m} = \frac{1}{x^m}$, we have

$$tang.\ \alpha = -\frac{m}{x^{m+1}} \text{ and } y - x\ tang.\ \alpha = x^{-m} + \frac{m}{x^m} = \frac{m+1}{x^m},$$

as also $y\ cotg.\ \alpha - x = -\frac{x}{m} - x = -(m+1)\frac{x}{m}$; hence:

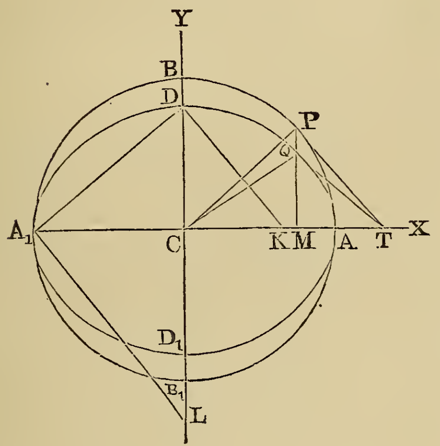
1. for $x = \infty$, $y = 0$, $tang.\ \alpha = 0$, $y - x\ tang.\ \alpha = 0$, and $n = 0$; and
2. for $y = \infty$, $x = 0$, $tang.\ \alpha = \infty$, $y\ cotg. - x = 0$, and $n = 0$.

But the axis of abscissas $X\overline{X}$ corresponds to the conditions $a = 0$ and $n = 0$, and the axis of ordinates $Y\overline{Y}$, to the conditions $a = \infty$ and $n = 0$; hence, these axes are, at the same time, asymptotes of the curves which correspond to the equation $y = x^{-m}$. (Comp. the curves $\overline{1}\,P_1\,\overline{1}$, $\overline{2}\,P_1\,\overline{2}$, and $\frac{\overline{1}}{2}\,P_1\,\frac{\overline{1}}{2}$, in Fig. 18, page 15.)

ART. 12. The equation of an *ellipse* $A\,D\,A_1\,D_1$, Fig. 20, may be immediately deduced from the equation

$$x^2 + y_1^2 = a^2$$

Fig. 20.



of the circle $A\,B\,A_1\,B_1$, having the radii $CA = CB = CP = a$, and the co-ordinates $C\,M = x$ and $MP = y_1$, if it be taken into consideration that the ordinate $M\,Q = y$ of the ellipse stands in the same relation to the ordinate $MP = y_1$ of the circle (the abscissas being the same), as the minor semi-axis $C\,D = b$ of the ellipse to the radius $C\,B = a$ of the circle. We have, therefore,

$$\frac{y}{y_1} = \frac{b}{a};\ \text{hence}\ y_1 = \frac{a}{b}\,y,\ \text{and}\ x^2 + \frac{a^2}{b^2}\,y^2 = a^2;$$

$$\text{i. e.:}\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ \text{as the equation of the ellipse.}$$

If, in this equation, we substitute $-\,b^2$ for $+\,b^2$, we obtain the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

of the *hyperbola* consisting of two branches $P\,A\,Q$ and $P_1\,A_1\,Q_1$, Fig. 21.

If, in the formula

$$y = \frac{b}{a}\,\sqrt{x^2 - a^2}$$

thus obtained, we take x infinitely great, a^2 will vanish in comparison with x^2, and

$$y = \frac{b}{a}\sqrt{x^2} = \pm\,\frac{b\,x}{a} = \pm\,x\ tang.\ a$$

will be the equation of two straight lines $C\,U$ and $C\,V$ passing through the origin C of the co-ordinates. As the ordinates

$$\pm\,\frac{b}{a}\,x = \frac{b}{a}\sqrt{x^2}\ \text{and}\ \frac{b}{a}\sqrt{x^2 - a^2}$$

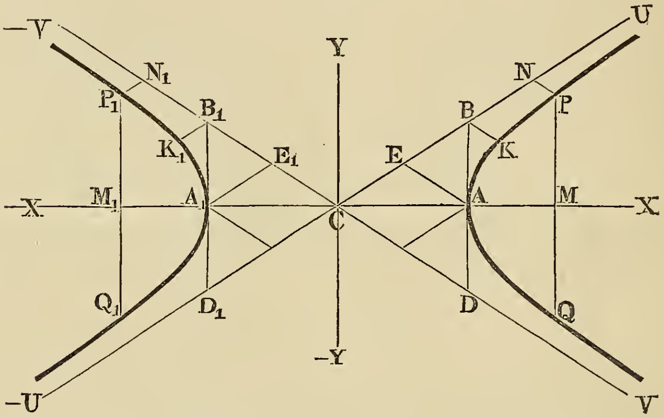
approach more and more to equality, the greater we assume x to be, it follows that the straight lines CU and CV are the *asymptotes of the hyperbola.*

If we take $CA = a$, as also the perpendiculars $AB = +b$ and $AD = -b$, we can thereby determine the two asymptotes; for we have, for the angles $\pm\alpha$ under which the axis of abscissas is intersected by the asymptotes:

$$tang.\ ACB = \frac{AB}{CA};\ \text{i. e.: } tang.\ \alpha = \frac{b}{a},\ \text{and likewise:}$$

$$tang.\ ACD = \frac{AD}{CA};\ \text{i. e.: } tang.\ (-\alpha) = -\frac{b}{a}.$$

Fig. 21.



If the asymptotes $U\overline{U}$ and $V\overline{V}$ be taken as co-ordinate axes, if, further, the abscissa or co-ordinate CN in the one axal direction be put $= u$, and the ordinate or co-ordinate NP in the other, $= v$, there will result, since the direction of u deviates from the axis of abscissas CX by the angle α, and that of v, from the same axis by the angle $-\alpha$, the abscissa:

$$CM = x = CN\ cos.\ \alpha + NP\ cos.\ \alpha = (u + v)\ cos.\ \alpha,$$

and the ordinate:

$$MP = y = CN\ sin.\ \alpha - NP\ sin.\ \alpha = (u - v)\ sin.\ \alpha.$$

If we further designate the hypothenuse $CB = \sqrt{a^2 + b^2}$ by e, we have:

$$cos.\ \alpha = \frac{a}{e} \text{ and } sin.\ \alpha = \frac{b}{e};$$

consequently: $\dfrac{cos.\ \alpha}{a} = \dfrac{sin.\ \alpha}{b} = \dfrac{1}{e}$ and

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{(u^2 + 2uv + v^2)}{a^2}\ cos.\ \alpha^2 - \frac{(u^2 - 2uv + v^2)}{b^2}\ sin.\ \alpha^2$$

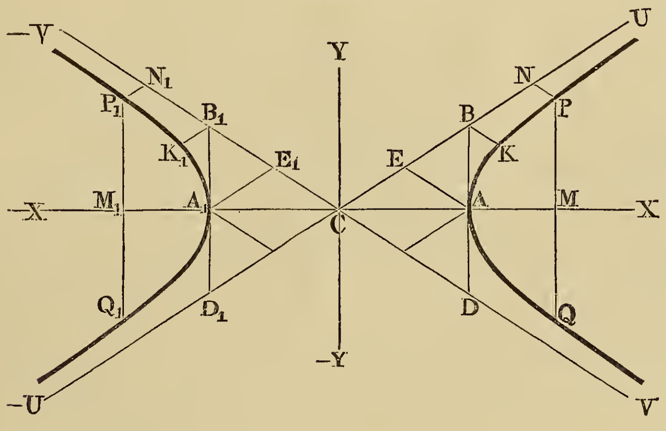
$$= \frac{u^2 + 2uv + v^2}{e^2} - \frac{u^2 - 2uv + v^2}{e^2} = \frac{4uv}{e^2} = 1,$$

from which there results the so-called *equation of asymptotes* of the hyperbola:

$$u\,v = \frac{e^2}{4}, \text{ or } v = \frac{e^2}{4u}.$$

It is, therefore, easy to draw the hyperbola between the given asymptotes. The co-ordinates for the vertex A are $CE = EA = \frac{e}{2}$, whilst

Fig. 22.



those for the point K are $CB = e$ and $BK = \frac{e}{4}$; we have, further, for the abscissas $2e$, $3e$, $4e$, &c, the ordinates $\frac{1}{2}\frac{e}{4}$, $\frac{1}{3}\frac{e}{4}$, $\frac{1}{4}\frac{e}{4}$, &c.

Art. 13. If, in the elementary ratio $\frac{\partial y}{\partial x}$, or in the formula for the tangent (*tang.* α, Art. 6,) of the tangential angle, we substitute successively different values for x, there will result the different positions of the line of contact of the appurtenant curve. If we take $x = 0$, we obtain the tangent of the tangential angle at the origin of the co-ordinates; but if we take $x = \infty$, we have the tangent for an infinitely distant point of the curve. The points where the tangent to a curve runs parallel with one of the axes of the co-ordinates are the most important; because, here, as a rule, the one or the other of the co-ordinates x and y has its *maximum* or *minimum* value; or is, as we say, a *maximum* or a *minimum*. For the parallelism with the axis of abscissas, we have $\alpha = 0$; therefore, also, *tang.* $\alpha = 0$; and for the parallelism with the axis of ordinates, $\alpha = 90^0$; therefore, *tang.* $\alpha = \infty$; and from this follows the rule: *The values of the abscissa or independent variable x, to which the maximum or minimum values of the ordinate or dependent variable y correspond, may be found by putting the differential ratio $\frac{\partial y}{\partial x} = 0$ and $= \infty$, and resolving the equations obtained, with respect to x.*

For the equation $y = 6x - \frac{9}{2}x^2 + x^3$, which corresponds to the curve $APQR$ in Fig. 23, we have, for example,

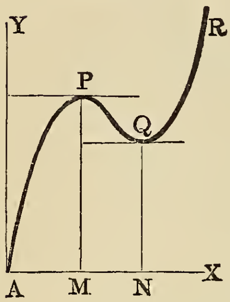
$$\frac{\partial y}{\partial x} = 6 - 9x + 3x^2 = 3(2 - 3x + x^2) = 3(1 - x)(2 - x);$$

and if we put $\frac{\partial y}{\partial x} = 0$, there results:

$$1 - x = 0 \text{ and } 2 - x = 0,$$

i. e.: $x = 1$ and $x = 2$.

Fig. 23.



If these values be put in the formula $y = 6x - \frac{9}{2}x^2 + x^3$, there will result the maximum value of y: $MP = 6 - \frac{9}{2} + 1 = \frac{5}{2}$, and the minimum value: $NQ = 12 - 18 + 8 = 2$.

We have, further, for the curve $KOPQR$ Fig. 24, whose equation is

$$y = x + \sqrt[3]{(x-1)^2},$$

$$\frac{\partial y}{\partial x} = tang\ \alpha = 1 + \tfrac{2}{3}(x-1)^{-\frac{1}{3}} = 1 + \frac{2}{3\sqrt[3]{x-1}};$$

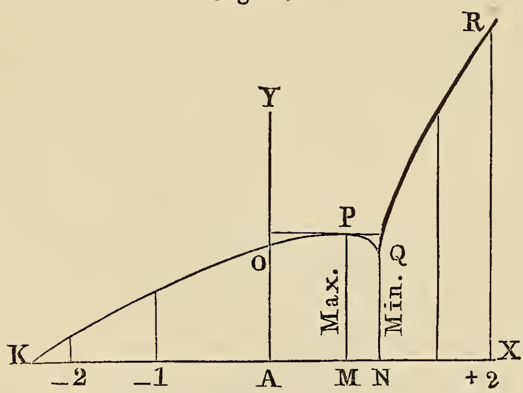
and, indeed, $= 0$ for $\frac{2}{3\sqrt[3]{x-1}} = -1$; i. e. for $AM = x = 1 - (\frac{2}{3})^3 = \frac{19}{27} = 0,7037$; on the other hand, $= \infty$ for $AN = x = 1$.

To the first case corresponds the *maximum value:*

$$MP = y_m = 1 - (\tfrac{2}{3})^3 + (\tfrac{2}{3})^2 = \tfrac{31}{27} = 1{,}148;$$

and to the last, the *minimum value:* $NQ = y_n = 1$.

Fig. 24.



There is also, further, for $x = 0$, $AO = y = 1$; on the other hand, $y = 0$ for the abscissa $AK = x$, which corresponds to the cubic equation $x^3 + x^2 - 2x + 1$, and has the value $x = -2, 148$.

ART. 14. In a *curve ascending* from the origin A, y increases with x, and ∂y is therefore positive, whilst in a *descending curve*, y decreases as x increases, and ∂y is, therefore, negative, and finally, zero, at the point where the curve runs parallel with the co-ordinate axis AX; likewise, the elements of ordinates:

$$SQ = PS\ tang.\ QPS, \text{ i. e. } \partial y_1 = \partial x \,.\, tang.\ \alpha_1,$$

$$TR = QT\ tang.\ RQT, \text{ i. e. } \partial y_2 = \partial x \,.\, tang.\ \alpha_2, \text{ \&c.,}$$

corresponding to like elements of abscissas $\partial x = MN = NO = PS = QT$ . . .

are, in a *convex curve* APR, Fig. 25, on the point of increase, whilst in a *concave curve* APR, Fig. 26, they are on the point of decrease. The same is also true of the tangential angles α_1, α_2, &c. Consequently, there is, in the first case:

$$\partial\ (tang.\ \alpha) = \partial \left(\frac{\partial y}{\partial x}\right) \textit{ positive,}$$

and in the second:

$$\partial\ (tang.\ \alpha) = \partial \left(\frac{\partial y}{\partial x}\right) \textit{ negative.}$$

Fig. 25.

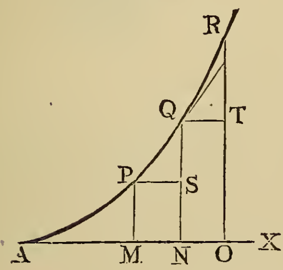


Fig. 26.

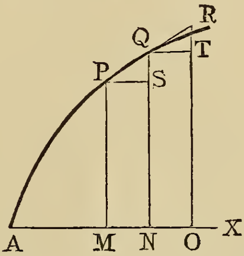
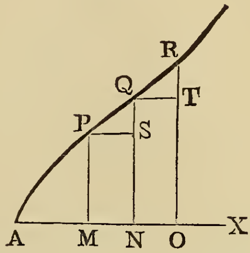


Fig. 27.



Lastly, we have for the *point of inflection* Q, Fig. 27, i. e. for that point of the curve where convexity passes into concavity, or *vice versa*, $SQ = TR$, and hence:

$$\partial\ (tang.\ \alpha) = \partial \left(\frac{\partial y}{\partial x}\right) = 0.$$

Therefore, the following rule is applicable: *If the differential of the tangent of the tangential angle be positive, the curve will be convex, if it be negative, the curve will be concave, and if it be zero, we shall have to consider a point of inflection of the curve.*

From the above, it is also easy to infer the following: The point where the curve runs parallel with the axis of abscissas, for which there is, therefore, $tang.\ \alpha = 0$, corresponds either to a *minimum*, or *maximum*, or to a *turning point* of the curve, according as this curve is *convex*, *concave*, or *neither one or the other*; i. e. according as we have:

$$\partial\ (tang.\ \alpha) \textit{ positive, negative,} \text{ or } \textit{zero.}$$

On the other hand, the point where a curve runs parallel with the axis of ordinates, for which we have $tang.\ \alpha = \infty$, corresponds to a *minimum*, *maximum*, or to a *turning point* of the curve, according as the same is *concave*, *convex*, or *partly concave* and *partly convex;* therefore, according as $\partial\ (tang.\ \alpha)$ is *negative* or *positive* before and after this point, or has a *sign* before it *different* from that which follows.

A portion of the curve with point of inflection Q, of the first kind, is shown in Fig. 28; and one of the second kind, in Fig. 29. We see the corresponding ordinate NQ is neither a maximum nor a

minimum; for in no case are the ordinates MP and OR simultaneously greater or less than NQ.

In geometry, physics, mechanics, &c., the finding of maximum and minimum values of a function is often of great importance. As, in the sequel, it will often be necessary to determine such values of functions, only the following geometrical problem will here be solved.

Fig. 28.

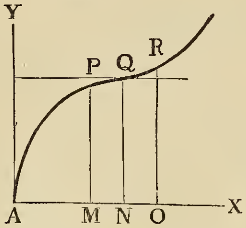


Fig. 29.

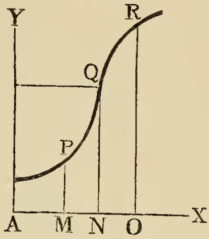
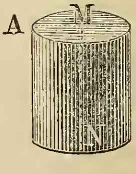


Fig. 30.



Required, the dimensions of a *right circular cylinder* AN, Fig. 30, which has, with a given content V, the smallest surface O. If the diameter of the base of this cylinder be designated by x, and the height of the same by y, we have:

$$V = \frac{\pi}{4} x^2 y,$$

and the surface, or the area of the two bases plus the area of the envelope:

$$O = \frac{2 \pi x^2}{4} + \pi x y;$$

or, since, according to the first equation, we may put

$$\pi y = \frac{4 V}{x^2}, \text{ therefore, } \pi x y = 4 V x^{-1}:$$

$$O = \frac{\pi x^2}{2} + 4 V x^{-1},$$

and consequently, as we can treat O and x as co-ordinates of a curve:

$$tang.\ \alpha = \frac{\partial O}{\partial x} = \pi x - 4 V x^{-2}.$$

If, now, we make this quotient zero, we obtain the equation of condition:

$$\pi x = \frac{4 V}{x^2}, \text{ or } \pi x^3 = 4 V,$$

the solution of which leads to

$$x = \sqrt[3]{\frac{4 V}{\pi}}, \text{ and}$$

$$y = \frac{4 V}{\pi x^2} = \sqrt[3]{\frac{64 V^3}{\pi^3} \cdot \frac{\pi^2}{16 V^2}} = \sqrt[3]{\frac{4 V}{\pi}} = x.$$

Since, further, $\partial\,(tang.\ \alpha) = \left(\pi + \frac{8 V}{x^3}\right) \partial x$ is *positive*, this result leads to the *minimum* sought.

This result is applicable also, when it is required to find the dimensions of a cylindrical vessel, which requires, with a given capacity, the least amount of material. It corresponds directly to this case, if the cover of the vessel is to be like the circular bottom; but if no cover is required, we have

$$O = \frac{\pi x^2}{4} + 4\, V x^{-1}, \text{ consequently:}$$

$$\frac{\pi x}{2} = \frac{4\, V}{x^2}, \text{ from which there follows:}$$

$$x = 2 \sqrt[3]{\frac{V}{\pi}}, \text{ and } y = \sqrt[3]{\frac{V^3}{\pi^3} \cdot \frac{\pi^2}{V^2}} = \sqrt[3]{\frac{V}{\pi}} = \tfrac{1}{2}\, x.$$

Whilst, therefore, in the first case, the *height of the cylinder* is to be taken *equal to its width*, in the second, the height is to be taken only *equal to one half of its width.*

ART. 15. By successive differentiation of a function $y = f(x)$, we find an entire series of new functions of the independent variable x, as:

$$f_1(x) = \frac{\partial y}{\partial x} = \frac{\partial f(x)}{\partial x},$$

$$f_2(x) = \frac{\partial f_1(x)}{\partial x},\ f_3(x) = \frac{\partial f_2(x)}{\partial x}, \text{ \&c.}$$

For $y = f(x) = x^{\frac{5}{3}}$, for example, there follows:

$f_1(x) = \frac{5}{3}\, x^{\frac{2}{3}}, f_2(x) = \frac{10}{9}\, x^{-\frac{1}{3}}, f_3(x) = -\frac{10}{27}\, x^{-\frac{4}{3}}$, &c.

For a function which is represented in a convergent series progressing according to the powers of x, whose exponents are positive whole numbers, as

$$y = f(x) = A_0 + A_1\, x + A_2\, x^2 + A_3\, x^3 + A_4\, x^4 + \cdots$$

we obtain

$$f_1(x) = A_1 + 2\, A_2\, x + 3\, A_3\, x^2 + 4\, A_4\, x^3 + \cdots$$
$$f_2(x) = 2\, A_2 + 2 \,.\, 3\, A_3\, x + 3 \,.\, 4\, A_4\, x^2 + \cdots$$
$$f_3(x) = 2 \,.\, 3\, A_3 + 2 \,.\, 3 \,.\, 4\, A_4\, x^2 + \cdots \text{ \&c.}$$

If, in these series, we make $x = 0$, we obtain expressions which may be used for finding the constant co-efficients $A_0, A_1, A_2 \ldots$ viz.:

$f(0) = A_0, f_1(0) = 1\, A_1, f_2(0) = 2\, A_2, f_3(0) = 2 \,.\, 3 \,.\, A_3$, &c., and hence, the co-efficients themselves:

$$A_0 = f(0),\ A_1 = f_1(0),\ A_2 = \tfrac{1}{2} f_2(0),\ A_3 = \frac{1}{2 \,.\, 3} f_3(0),$$

$$A_4 = \frac{1}{2 \,.\, 3 \,.\, 4} f_4(0), \text{ \&c.}$$

Accordingly, a function may be developed into the following form, termed *Maclaurin's series:*

$$f(x) = f(0) + f_1(0) \cdot \frac{x}{1} + f_2(0) \cdot \frac{x^2}{1.2} + f_3(0) \cdot \frac{x^3}{1.2.3} + f_4(0) \cdot \frac{x^4}{1.2.3.4} + \cdots$$

For the *binomial function* $y = f(x) = (1 + x)^n$, we have

$$f_1(x) = n(1+x)^{n-1}, f_2(x) = n(n-1)(1+x)^{n-2},$$
$$f_3(x) = n(n-1)(n-2)(1+x)^{n-3}, \text{ \&c.};$$

therefore, if we put $x = 0$, there results:

$$f(0) = 1, f_1(0) = n, f_2(0) = n(n-1),$$
$$f_3(0) = n(n-1)(n-2), \text{ \&c.},$$

and there follows the *binomial series:*

$$\text{I.} \quad (1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \cdots \text{ \&c.}$$

We have further:

$$(1 = x)^n = 1 - \frac{n}{1}x + \frac{n(n-1)}{1.2}x^2 - \frac{n(n-1)(n-2)}{1.2.3}x^3 + \cdots$$

as also:

$$(1+x)^{-n} = 1 - \frac{n}{1}x + \frac{n(n+1)}{1.2}x^2 - \frac{n(n+1)(n+2)}{1.2.3}x^3 + \cdots$$

If we put $1 + x = (1-z)^{-1} = \frac{1}{1-z}$, there follows $z = \frac{x}{1+x}$, and

$$(1+x)^n = (1-z)^{-n} = 1 + nz + \frac{n(n+1)}{1.2}z^2 + \frac{n(n+1)(n+2)}{1.2.3}z^3 + \cdots \text{ i. e.:}$$

$$\text{II.} \quad (1+x)^n = 1 + \frac{n}{1}\left(\frac{x}{1+x}\right) + \frac{n(n+1)}{1.2}\left(\frac{x}{1+x}\right)^2 + \frac{n(n+1)(n+2)}{1.2.3}\left(\frac{x}{1+x}\right)^3 + \cdots$$

The series under I. is a finite one for entire positive values of n, and that under II. is a finite one for entire negative values of n. For example,

$$(1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5, \text{ and}$$

$$(1+x)^{-5} = 1 - 5\left(\frac{x}{1+x}\right) + 10\left(\frac{x}{1+x}\right)^2 - 10\left(\frac{x}{1+x}\right)^3 + 5\left(\frac{x}{1+x}\right)^4 - \left(\frac{x}{1+x}\right)^5.$$

Since we have $a + x = a\left(1 + \frac{x}{a}\right)$, there follows also:

$$(a+x)^n = a^n \left(1+\frac{x}{a}\right)^n = a^n \left[1 + \frac{n}{1}\left(\frac{x}{a}\right) + \frac{n(n-1)}{1\,.\,2}\left(\frac{x}{a}\right)^2 + \cdots\right], \text{ i. e.:}$$

III. $$(a+x)^n = a^n + \frac{n}{1} a^{n-1} x + \frac{n(n-1)}{1\,.\,2} a^{n-2} x^2 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3} a^{n-3} x^3 + \cdots$$

For example: $\sqrt[3]{1009^2} = (1000+9)^{\frac{2}{3}} = 100\,(1+0{,}009)^{\frac{2}{3}}$

$$= 100\left(1 + \tfrac{2}{3}\,.\,0{,}009 + \frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2}\,.\,(0{,}009)^2 + \cdots\right)$$

$$= 100\,(1 + 0{,}006 - 0{,}000009) = 100{,}5991.$$

There is also:

$$(x+1)^n = x^n + n x^{n-1} + \frac{n(n-1)}{1\,.\,2} x^{n-2} + \cdots$$

hence, for very great values of x, approximately:

$$(x+1)^n = x^n + n x^{n-1}.$$

Accordingly, there follows $x^{n-1} = \dfrac{(x+1)^n - x^n}{n}$; further:

$$(x-1)^{n-1} = \frac{x^n - (x-1)^n}{n},$$

$$(x-2)^{n-1} = \frac{(x-1)^n - (x-2)^n}{n},$$

$$(x-3)^{n-1} = \frac{(x_2-2)^n - (x-3)^n}{n},$$

$$\vdots \qquad = \qquad \vdots$$

and lastly: $$1^{n-1} = \frac{2^n - 1^n}{n}.$$

By adding to both sides of the equation, there follows now:

$$x^{n-1} + (x-1)^{n-1} + (x-2)^{n-1} + (x-3)^{n-1} + \cdots + 1 = \frac{(x+1)^n - 1^n}{n};$$

or if we put $n - 1 = m$, therefore, $n = m + 1$, and invert the series:

$$1^m + 2^m + 3^m + \cdots + (x-1)^m + x^m = \frac{(x+1)^{m+1} - 1}{m+1}.$$

Further, since x should be infinitely great, we may put $(x+1)^{m+1} = x^{m+1}$; whence there follows the *sum of the powers of the natural progression of numbers:*

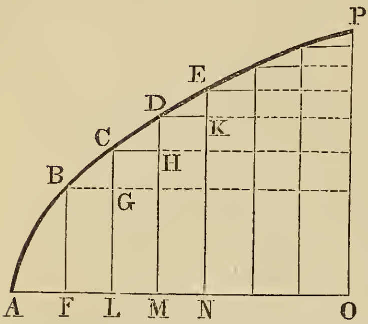
IV. $$1^m + 2^m + 3^m + \cdots + x^m = \frac{x^{m+1}}{m+1};$$ for example:

$\sqrt[3]{1^2} + \sqrt[3]{2^2} + \sqrt[3]{3^2} + \sqrt[3]{4^2} + \cdots + \sqrt[3]{1000^2}$, approximately

$$= \frac{1000^{\frac{5}{3}}}{\frac{5}{3}} = \tfrac{3}{5}\sqrt[3]{1000^5} = 60000.$$

ART. 16. The ordinate $OP = y$, corresponding to the abscissa $AO = x$, Fig. 31, may be composed of an infinite number of unequal elements ∂y, as FB, GC, HD, KE . . . which correspond to the equal elements $\partial x = AF = FL = LM = MN$ . . . of the abscissa. If, therefore, we had given $\partial y = \varphi(x) \,.\, \partial x$, y would be found by summation of all those values of ∂y, which result, when, in $\varphi(x) \,.\, \partial x$, we substitute for x successively ∂x, $2\partial x$, $3\partial x$, $4\partial x$ . . . up to $n\partial x = x$.

Fig. 31.



This summation is indicated by the *integral sign* $\int$ which is placed before the general expression for the elements which are to be summed up; thus, instead of

$$y = [\varphi(\partial x) + \varphi(2\partial x) + \varphi(3\partial x) + \ldots + \varphi(x)]\,\partial x,$$

we write $y = \int \varphi(x)\,\partial x$.

In this case y is called the *integral* of $\varphi(x)\,\partial x$, as also $\varphi(x)\,\partial x$, the *differential* of y.

Oftentimes, the integral $\int \varphi(x)\,\partial x$ may be determined by actual summation of the series $\varphi(\partial x)$, $\varphi(2\partial x)$, $\varphi(3\partial x)$, &c.; it is, however, much more simple to make use of one of the rules of the *integral calculus*, which are to be developed in the sequel.

If n be the number of the elements (∂x) of x, therefore, $x = n\partial x$, or $\partial x = \frac{x}{n}$, we may put:

$$\int \varphi(x)\,\partial x = \left[\varphi\left(\frac{x}{n}\right) + \varphi\left(\frac{2x}{n}\right) + \varphi\left(\frac{3x}{n}\right) + \cdots + \varphi\left(\frac{nx}{n}\right)\right]\frac{x}{n}.$$

For the differential $\partial y = ax\partial x$, we have, for instance, the integral:

$$y = \int ax\partial x = a\partial x\,(\partial x + 2\partial x + 3\partial x + \cdots + n\partial x)$$
$$= (1 + 2 + 3 + \cdots + n)\,a\partial x^2;$$

or, since, according to Art. 15, IV., we have for $n = \infty$, the sum of the natural progression of numbers $1 + 2 + 3 + \ldots + n = \frac{1}{2}n^2$, and $\partial x^2 = \frac{x^2}{n^2}$,

$$y = \int ax\partial x = \tfrac{1}{2}\,n^2\,a\,\frac{x^2}{n^2} = \tfrac{1}{2}\,ax^2.$$

In a similar manner we find:

$$y = \int \varphi(x)\,\partial x = \int \frac{x^2\partial x}{a} = [(\partial x)^2 + (2\partial x)^2 + (3\partial x)^2 + \cdots + (n\partial x)^2]\frac{\partial x}{a}$$
$$= (1 + 2^2 + 3^2 + \cdots + n^2)\,\frac{\partial x^3}{a},$$

if we put $x = n\partial x$, or assume it to consist of n elements ∂x.

But according to § 15, IV., we have, for $n = \infty$,

$$1 + 2^2 + 3^2 + \cdots + n^2 = \frac{n^2}{3}; \text{ hence there follows:}$$

$$\int \frac{x^2 \partial x}{a} = \frac{n^3}{3} \cdot \frac{\partial x^3}{a} = \frac{(n \partial x)^3}{3a} = \frac{x^3}{3a}.$$

ART. 17. From the formula $\partial [a + mf(x)] = m \partial f(x)$, we have, by inversion, $\int m \partial f(x) = a + mf(x) = a + m \int \partial f(x)$; or, if we put
$$\partial f(x) = \varphi(x) \,.\, \partial x:$$
$$\text{I.} \quad \int m \varphi(x) \partial x = a + m \int \varphi(x) \partial x;$$
and hence it follows: that the *constant factor m remains unchanged* by integration as well as by differentiation; that an existing *constant member a* cannot be determined by integration alone; and that, therefore, mere integration produces an integral which is still *undetermined.*

To find the constant member, two associated values of x and $y = \int \varphi(x) \partial x$ must be known. If we have for $x = c$, $y = k$, and if we have found that $y = \int \varphi(x) \partial x = a + f(x)$, there must be also:
$$k = a + f(c),$$
and the subtraction gives: $y - k = f(x) - f(c)$; therefore, in this case:
$$y = \int \varphi(x) \partial x = k + f(x) - f(c) = f(x) + k - f(c);$$
and we have, accordingly, the constant $a = k - f(c)$.

If it be known, for example, that the undetermined integral

$$y = \int x \partial x = \frac{x^2}{2} \text{ gives } y = 3 \text{ for } x = 1, \text{ we have the required}$$

constant $a = 3 - \frac{1}{2} = \frac{5}{2}$; and hence, the integral:
$$y = \int x \partial x = a + \frac{x^2}{2} = \frac{5 + x^2}{2}.$$

Even the determination of the constant, leaves the integral still undetermined; since, for x as independent variable, any arbitrary value may be assumed; but if we require an entirely determinate value k_1 of the integral, which corresponds to a determined value c_1 of x, the latter must be, further, introduced in the integral already found; therefore $k_1 = k + f(c_1) - f(c)$.

Thus, for $x = 5$, $y = \int x \partial x = \frac{5 + x^2}{2}$ gives $y = 15$.

That value of x for which $y = 0$, is the best known; in this case, therefore, we have $k = 0$, and hence the indeterminate integral $\int \varphi(x) \partial x = f(x)$ leads to the determinate integral $k_1 = f(c_1) - (c)$, which is found, if, in the expression $f(x)$ for the former, the two given limiting values c_1 and c of x be introduced, and the ascertained values be subtracted the one from the other.

To indicate this, we write, instead of $\int \varphi(x)\,\partial x$, $\int_c^{c_1} \varphi(x)\,\partial x$; if, therefore, we have $\int \varphi(x)\,\partial x = \frac{x^2}{2}$, $\int_c^{c_1} \varphi(x)\,\partial x = \frac{c_1^{\,2} - c^2}{2}$.

The inversion of the differential formula $\partial[f(x) + \varphi(x)] = \partial f(x) + \partial \varphi(x)$, gives the integral formula: $\int[\partial f(x) + \partial \varphi(x)] = f(x) + \varphi(x)$; or, if we put $\partial f(x) = \psi(x)\,\partial x$, and $\partial \varphi(x) = \chi(x)\,\partial x$:

$$\text{II.}\quad \int[\psi(x)\,\partial x + \chi(x)\,\partial x] = \int \psi(x)\,\partial x + \int \chi(x)\,\partial x.$$

Therefore, *the integral of a sum of several differentials is equal to the sum of the integrals of the single differentials.*

For example, $\int(3 + 5x)\,\partial x = \int 3\,\partial x + \int 5\,x\,\partial x = 3x + \frac{5}{2}x^2$.

ART. 18. The most important differential formula IV. of Art. 8:

$$\partial(x^n) = n x^{n-1}\,\partial x,$$

leads, by inversion, to the equally important integral formula. There is, accordingly, $\int n x^{n-1}\,\partial x = x^n$, or $n\int x^{n-1}\,\partial x = x^n$, hence:

$$\int x^{n-1}\,\partial x = \frac{x^n}{n};$$

if, therefore, we put $n - 1 = m$, and consequently, $n = m + 1$, we obtain the following important *integral:*

$$\int x^m\,\partial x = \frac{x^{m+1}}{m+1},$$

which occurs in practice as often, at least, as all the others together.

The form of this integral indicates also that it corresponds to the system of curves treated of in Art. 9, and illustrated in Fig. 17.

From this we have, for example,

$$\int 5\,x^3\,\partial x = 5\int x^3\,\partial x = \tfrac{5}{4}\,x^4;\ \text{further,}$$

$$\int \sqrt[3]{x^4}\,\partial x = \int x^{\frac{4}{3}}\,\partial x = \tfrac{3}{7}\,x^{\frac{7}{3}} = \tfrac{3}{7}\,\sqrt[3]{x^7};$$

$$\int \frac{\partial x}{2\sqrt{x}} = \tfrac{1}{2}\int x^{-\frac{1}{2}}\,\partial x = \tfrac{1}{2}\,\frac{x^{\frac{1}{2}}}{\frac{1}{2}} = \sqrt{x};$$

$$\int(4 - 6\,x^2 + 5\,x^4)\,\partial x = \int 4\,\partial x - \int 6\,x^2\,\partial x + \int 5\,x^4\,\partial x$$
$$= 4\int \partial x - 6\int x^2\,\partial x + 5\int x^4\,\partial x = 4\,x - 2\,x^3 + x^5;$$

and, if we introduce $3\,x - 2 = u$, therefore, $3\,\partial x = \partial u$, or $\partial x = \frac{\partial u}{3}$:

$$\int \sqrt{3\,x - 2}\,.\,\partial x = \int u^{\frac{1}{2}}\frac{\partial u}{3} = \tfrac{1}{3}\,\frac{u^{\frac{3}{2}}}{\frac{3}{2}} = \tfrac{2}{9}\,\sqrt{u^3}$$
$$= \tfrac{2}{9}\,\sqrt{(3\,x - 2)^3};$$

lastly, if we put $2\,x^2 - 1 = u$, therefore, $4\,x\,\partial x = \partial u$, or $x\,\partial x = \frac{\partial u}{4}$:

$$\int \frac{5\,x\,\partial x}{\sqrt[3]{2x^2-1}} = \int \frac{5\,\partial u}{4\,\sqrt[3]{u}} = \frac{5}{4}\int u^{-\frac{1}{3}}\,\partial u = \frac{5}{4}\,\frac{u^{\frac{2}{3}}}{\frac{2}{3}}$$

$$= \frac{15}{8}\sqrt[3]{u^2} = \frac{15}{8}\sqrt[3]{(2\,x^2-1)^2}.$$

By appending the limiting values, the indeterminate integrals are immediately changed to determinate ones; for example,

$$\int_1^2 5\,x^3\,\partial x = \tfrac{5}{4}\,(2^4 - 1^4) = \tfrac{5}{4}\,.\,(16-1) = 18\tfrac{3}{4},$$

$$\int_4^9 \frac{\partial x}{2\,\sqrt{x}} = \sqrt{9} - \sqrt{4} = 1,$$

$$\int_1^{16} \sqrt{3\,x-2}\,.\,\partial x = \tfrac{2}{9}\,(\sqrt{16^3} - \sqrt{1^3}) = \tfrac{2}{9}\,(64-1) = 14.$$

If, for instance, we had for $x = 0$, $\int (4 - 6x^2 + 5\,x^4)\,\partial x = 7$, we should have generally:

$$\int (4 - 6\,x^2 + 5\,x^4)\,\partial x = 7 + 4\,x - 2\,x^3 + x^5.$$

Art. 19. The so-called *exponential function* $y = a^x$, which consists of a power with variable exponents, may, by aid of *Maclaurin's Theorem*, be developed into a series, in which process, the corresponding differential will also be found.

If we put $a^x = A_0 + A_1\,x + A_2\,x^2 + A_3\,x^3 + \ldots$, or, (since for $x=0$, a^x assumes the value $a^0 = 1$, and we have, therefore, $A_0 = 1$,)

$$a^x = 1 + A_1\,x + A_2\,x^2 + A_3\,x^3 + \ldots, \text{ we have also:}$$

$$a^{\partial x} = 1 + A_1\,\partial x + A_2\,\partial x^2 + A_3\,\partial x^3 + \ldots, \text{ and hence:}$$

$$\partial\,(a^x) = a^{x+\partial x} - a^x = a^x\,a_{\partial x} - a^x = a^x\,(a^{\partial x} - 1)$$

$$= a^x\,(A_1\,\partial x + A_2\,\partial x^2 + A_3\,\partial x^3 + \ldots)$$

$$= a^x\,(A_1 + A_2\,\partial x + \ldots)\,\partial x = A_1\,a^x\partial x.$$

Now, from successive differentiation of the series, there follows:

$$f\,(x) = a^x = 1 + A_1\,x + A_2\,x^2 + A_3\,x^3 + \cdots,$$

$$f_1\,(x) = \frac{\partial\,(a^x)}{\partial x} = A_1\,a^x = A_1 + 2\,A_2\,x + 3\,A_3\,x^2 + \cdots,$$

$$f_2\,(x) = \frac{\partial\,(A_1\,a^x)}{\partial x} = A_1^2\,a^x = 2\,A_2 + 2\,.\,3\,.\,A_3\,x + \cdots,$$

$$f_3\,(x) = \frac{\partial\,(A_1^2\,a^x)}{\partial x} = A_1^3\,a^x = 2\,.\,3\,.\,A_3 + \cdots,$$

hence, if we put $x = 0$, there results:

$$A_1 = A_1,\ 2\,A_2 = A_1^2,\ 2\,.\,3\,.\,A_3 = A_1^3 + \cdots, \text{ whence,}$$

$$A_2 = \frac{1}{1\,.\,2}\,A_1^2,\ A_3 = \frac{1}{1\,.\,2\,.\,3}\,A_1^3,\ A_4 = \frac{1}{1\,.\,2\,.\,3\,.\,4}A_1^4, \text{\&c.,}$$

and the exponential series takes the form:

$$\text{I.}\quad a^x = 1 + A_1\,\frac{x}{1} + A_1^2\,\frac{x^2}{1\,.\,2} + A_1^3\,\frac{x^3}{1\,.\,2\,.\,3}$$

$$+ A_1^4\,\frac{x^4}{1\,.\,2\,.\,3\,.\,4} + \cdots$$

The constant co-efficient A_1 is, of course, a determinate function of the constant base, as also, the latter is a function of the former; hence, if one of the two numbers be given, the other is thereby determined. The simplest, or so-called *natural*, *series of powers*, is obtained for $A_1 = 1$, whose base (a) is, in the sequel, to be indicated by e. There is, therefore,

$$\text{II.} \quad e^x = 1 + \frac{x}{1} + \frac{x^2}{1 \,.\, 2} + \frac{x^3}{1 \,.\, 2 \,.\, 3} + \frac{x^4}{1 \,.\, 2 \,.\, 3 \,.\, 4} + \cdots,$$

and if we put $x = 1$, there will result the *base* of the natural series of powers:

$$e^1 = e = 1 + 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} + \cdots = 2{,}7182828 \cdots$$

If we put $e = a^m$, or $a = e^{\frac{1}{m}}$, we have $\frac{1}{m} = nat.\ log.\ a$, the so-called *natural*, or *hyperbolic logarithm* of a, and

$$\text{III.} \quad a^x = (e^{\frac{1}{m}})^x = e^{\frac{x}{m}} = 1 + \frac{1}{1}\left(\frac{x}{m}\right) + \frac{1}{1 \,.\, 2}\left(\frac{x}{m}\right)^2 + \frac{1}{1 \,.\, 2 \,.\, 3}\left(\frac{x}{m}\right)^3 + \cdots$$

Since this series coincides in form with that under I., we have also $A_1 = \dfrac{1}{m}$, and

$$\text{IV.} \quad \partial\,(a^x) = A_1\, a^x\, \partial x = \frac{a^x\, \partial x}{m} = nat.\ log.\ a\ .\ a^x \partial x, \text{ as also:}$$

$$\text{V.} \quad \partial\,(e^x) = e^x\, \partial x.$$

For example: $\partial\,(e^{3x+1}) = e^{3x+1}\, \partial\,(3\,x + 1) = 3\, e^{3x+1}\, \partial x.$

If we put $y = a^x = e^{\frac{x}{m}}$, we have inversely:

$$x = log._a\, y \text{ and } \frac{x}{m} = nat.\ log.\ y, \text{ hence:}$$

$$log._a\, y = m\ nat.\ log.\ y, \text{ and inversely,}$$

$$nat.\ log.\ y, \text{ or } log._e\, y = \frac{1}{m}\ log._a\, y.$$

The number m is called the *modulus* of the system of logarithms corresponding to the base a. Therefore, by help of the same, the natural logarithm may be changed into any artificial logarithm, and *vice versa*. For *Briggs'* system of logarithms, the base is $a = 10$, hence $\frac{1}{m} = nat.\ log.\ 10 = 2{,}30258\ \ldots$, and inversely, the modulus:

$$m = \frac{1}{nat.\ log.\ 10} = 0{,}43429\ \ldots,$$

therefore,

$$log.\ y = 0{,}43429\ nat.\ log.\ y, \text{ and}$$

$$nat.\ log.\ y = 2{,}30258\ log.\ y.$$

Art. 20. The course of the curves which correspond to the exponential functions $y = e^x$ and $y = 10^x$, is illustrated in Fig. 32.

For $x = 0$, we have in both cases $y = e^0 = a^0 = 1$; hence, also, both curves OQS and OQ_1S_1 pass through the same point (O) in the axis of ordinates AY. For $x = 1$, we have

$$y = e^x = 2{,}718 \ldots \text{ and}$$
$$y = 10^x = 10,$$

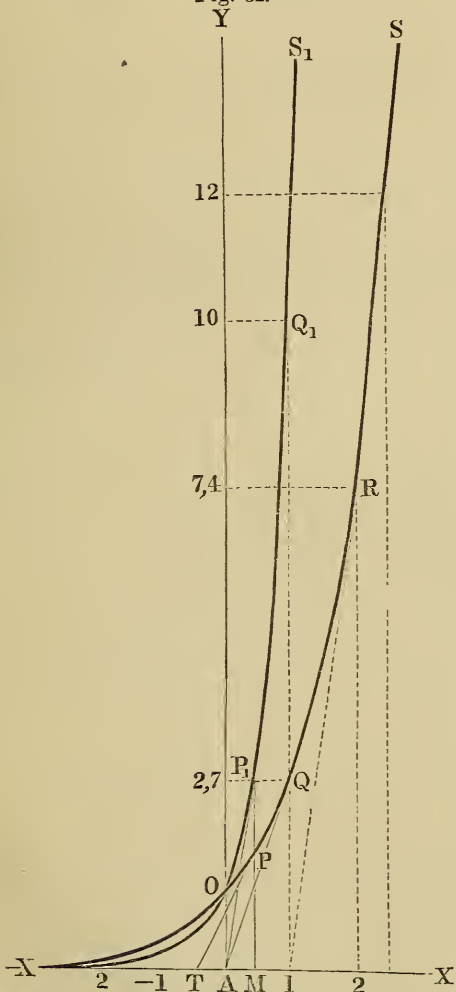
for $x = 2$,

$$y = e^x = 2{,}718^2 = 7{,}389 \text{ and}$$
$$y = 10^x = 10^2 = 100, \text{ \&c.};$$

therefore, on the positive side of the axis of abscissas, both curves ascend very perpendicularly; and particularly the last. On the other hand we have, for $x = -1$:

$$e^x = e^{-1} = \frac{1}{2{,}718 \ldots} = 0{,}368 \ldots \text{ and}$$
$$10^x = 10^{-1} = 0{,}1;$$

Fig. 32.



further, for $x = -2$:

$$e^x = e^{-2} = \frac{1}{2{,}718^2} = 0{,}135 \text{ and } 10^x = 10^{-2} = 0{,}01;$$

and, lastly, for $x = -\infty$, both equations give:

$$e^{-\infty} = \frac{1}{e^\infty} = \frac{1}{a^\infty} = 0.$$

Consequently, on the negative side of the axis of abscissas, the two curves approach nearer and nearer to the same, the latter, indeed, more abruptly than the former, but an actual coincidence with the axis never takes place.

As from $y = e^x$, there results $x = nat.\ log.\ y$, and likewise from $y = a^x$, $x = log._a\ y$, these curves furnish a scale of the natural logarithms, and of those of *Briggs;* the abscissas are, viz., the logarithms of the ordinates, and we have, for example,

$$AM = nat.\ log.\ MP$$
$$= log._a\ MP_1, \text{ \&c.}$$

According to the differential formula IV. of the last article, the tangential angle α of the exponential curve is determined by the simple formula:

$$tang.\ \alpha = \frac{\partial y}{\partial x} = \frac{a^x \partial x}{m\, \partial x} = \frac{a^x}{m} = \frac{y}{m} = y\ nat.\ log.\ a.$$

In the curve $O P_1 Q_1 S_1$, Fig. 32, the *subtangent* is, consequently, $= y\ cotg.\ a = m$, therefore, constant; and in the curve $O P Q S$, it is always $= 1$; for example, for the point Q, $\overline{A\,1} = 1$; for the point R, $\overline{12} = 1$, &c.

Art. 21. If there be $x = a^y$, there is further:

$$\partial x = \partial (a^y) = \frac{a^y\, \partial y}{m},$$

and inversely:

$$\partial y = \frac{m\, \partial x}{a^y} = \frac{m\, \partial x}{x}.$$

But we have also $y = log._a\, x$; i. e. the logarithm of the variable power x with the constant base a, whence there results the following differential formulae of the *logarithmic functions*

$$y = log._a\, x \text{ and } y = nat.\ log.\ x:$$

$$\text{I.}\quad \partial\, (log._a\, x) = \frac{m\, \partial x}{x} = \frac{1}{nat.\ log.\ a} \cdot \frac{\partial x}{x}, \text{ as also,}$$

$$\text{II.}\quad \partial\, (nat.\ log.\ x) = \frac{\partial x}{x}.$$

If a be the tangential angle of the curve which corresponds to the equation $y = log._a\, x$, we have also $tang.\ a = \frac{m}{x}$, and the *subtangent*,

$$= y\ cotg.\ a = \frac{xy}{m};$$

therefore, proportional to the area xy of a rectangle to be constructed from the sides x and y.

By means of the differential formulae I. and II. we obtain:

$$1.\ \partial\, (nat.\ log.\ \sqrt[2]{x}) = \frac{\partial \sqrt[2]{x}}{\sqrt{x}} = \frac{\partial\, (x^{\frac{1}{2}})}{x^{\frac{1}{2}}} = \tfrac{1}{2}\, \frac{x^{-\frac{1}{2}}\, \partial x}{x^{\frac{1}{2}}} = \frac{\partial x}{2x},$$

$$\text{or also} = \partial\, (\tfrac{1}{2}\ nat.\ log.\ x) = \tfrac{1}{2}\, \partial\, (nat.\ log.\ x) = \tfrac{1}{2} \cdot \frac{\partial x}{x}.$$

$$\begin{aligned} 2.\quad \partial\ nat.\ log.\left(\frac{2 + x}{x^2}\right) &= \partial\, [log.\ (2 + x) - log.\ x^2] \\ &= \partial\ log.\ (2 + x) - \partial\ log.\ (x^2) \\ &= \frac{\partial x}{2 + x} - 2\,\frac{\partial x}{x} = -\,\frac{(4 + x)\, \partial x}{x\, (2 + x)}. \end{aligned}$$

$$3.\quad \partial \left(nat.\ log.\ \frac{e^x - 1}{e^x + 1}\right) = \partial\, [nat.\ log.\ (e^x - 1)] - \partial\, [nat.\ log.\ (e^x + 1)]$$

$$= \frac{\partial\, (e^x)}{e^x - 1} - \frac{\partial\, (e^x)}{e^x + 1} = \frac{e^x\, \partial x}{e^x - 1} - \frac{e^x\, \partial x}{e^x + 1} = \frac{2\, e^x\, \partial x}{e^{2x} - 1}.$$

Art. 22. If the differential formulae of the foregoing article become inverted, we meet with other important integral formulae as follows.

From $\partial (a^x) = \frac{a^x \partial x}{m}$, there follows $\int \frac{a^x \partial x}{m} = a^x$, i. e.:

I. $\int a^x \partial x = m a^x = a^x : nat.\ log.\ a$, and hence:

II. $\int e^x \partial x = e^x$.

Further, from $\partial (log._a x) = \frac{m \partial x}{x}$, there follows $\int \frac{m \partial x}{x} = log._a x$, i. e.:

III. $\int \frac{\partial x}{x} = \frac{1}{m} log._a x = nat.\ log.\ x$, and the formula $\partial (nat.\ log.\ x) = \frac{\partial x}{x}$ gives the same.

From this, the following examples may be easily performed:

$$\int e^{5x-1} \partial x = \tfrac{1}{5} \int e^{5x-1} \partial (5x - 1) = \tfrac{1}{5} e^{5x-1}.$$

$$\int \frac{3 \partial x}{7x + 2} = \tfrac{3}{7} \int \frac{\partial (7x + 2)}{7x + 2} = \tfrac{3}{7}\ nat.\ log.\ (7x + 2).$$

$$\int \left(\frac{x^2 + 1}{x - 1}\right) \partial x = \int \left(x + 1 + \frac{2}{x - 1}\right) \partial x$$

$$= \int x \partial x + \int \partial x + 2 \int \frac{\partial (x - 1)}{x - 1} = \frac{x^2}{2} + x + 2\ nat.\ log.\ (x - 1).$$

Art. 23. The first integral formula $\int x^m \partial x = \frac{x^{m+1}}{m + 1}$, leaves the last integral undetermined; for if we put $m = -1$, there follows: $\int \frac{\partial x}{x} = \int x^{-1} \partial x = \frac{x^0}{0} +$ a constant $= \infty +$ constant; but if we put $x = 1 + u$, and $\partial x = \partial u$, we obtain:

$$\frac{\partial x}{x} = \frac{\partial u}{1 + u} = (1 - u + u^2 - u^3 + u^4 - \cdots) \partial u, \text{ and hence,}$$

$$\int \frac{\partial x}{x} = \int \frac{\partial u}{1 + u} = \int (1 - u + u^2 - u^3 + u^4 - \cdots) \partial u$$

$$= \int \partial u - \int u \partial u + \int u^2 \partial u - \int u^3 \partial u + \cdots$$

$$= u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \cdots;$$

we may, therefore, also put $nat.\ log.\ (1 + u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \cdots$, or:

IV. $Nat.\ log.\ x = (x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \frac{(x - 1)^4}{4} + \ldots$

By the aid of this series, we may also calculate the logarithms of those numbers which deviate merely a trifle from 1; but if it be required to find the logarithms of greater numbers, we proceed as follows.

If u be taken negative, the series before the last gives

$$nat.\ log.\ (1 - u) = - u - \frac{u^2}{2} - \frac{u^3}{3} - \frac{u^4}{4} - \cdots;$$

and, by subtraction of one series from the other, there results

$$nat.\ log.\ (1+u) - nat.\ log.\ (1-u) = 2\left(u + \frac{u^3}{3} + \frac{u^5}{5} + \cdots\right), \text{ i. e.}$$

$$nat.\ log.\ \left(\frac{1+u}{1-u}\right) = 2\left(u + \frac{u^3}{3} + \frac{u^5}{5} + \cdots\right);$$ or, if we make

$$\frac{1+u}{1-u} = x, \text{ therefore } u = \frac{x-1}{x+1},$$

V. $$Nat.\ log.\ x = 2\left[\frac{x-1}{x+1} + \tfrac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \tfrac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \cdots\right].$$

This series may also be employed for determining the logarithms of such numbers as deviate considerably from 1, since $\frac{x-1}{x+1}$ is always less than 1.

Therefore, also, $log.\ (x+y) - log.\ x = log.\ \left(\frac{x+y}{x}\right) = log.\left(1+\frac{y}{x}\right)$

$$= \frac{y}{x} - \tfrac{1}{2}\left(\frac{y}{x}\right)^2 + \tfrac{1}{3}\left(\frac{y}{x}\right)^3 - \text{\&c.}$$

$$= 2\left[\frac{y}{2x+y} + \tfrac{1}{3}\left(\frac{y}{2x+y}\right)^3 + \tfrac{1}{5}\left(\frac{y}{2x+y}\right)^5 + \cdots\right],$$

and hence:

VI. $$Log.\ (x+y) = log.\ x + 2\left[\frac{y}{2x+y} + \tfrac{1}{3}\left(\frac{y}{2x+y}\right)^3 + \cdots\right].$$

This formula may also be used to find, from one logarithm, the next greater. For example,

$$nat.\ log.\ 2 = 2\left[\frac{2-1}{2+1} + \tfrac{1}{3}\cdot\left(\frac{2-1}{2+1}\right)^3 + \cdots\right]$$

$$= 2\left(\tfrac{1}{3} + \tfrac{1}{3}\cdot\tfrac{1}{27} + \tfrac{1}{5}\cdot\tfrac{1}{243} + \cdots\right)$$

$$= 2\left\{\begin{matrix} 0{,}33333 \\ 0{,}01234 \\ 0{,}00082 \\ 0{,}00007 \end{matrix}\right\} = 2\,.\,0{,}34656 = 0{,}69312,$$

more correctly, $= 0{,}69314718.$

Nat. log. $8 =$ *nat. log.* $2^3 = 3$ *nat. log.* 2 is, accordingly, $= 2{,}0794415$, and, finally, from the last formula:

$$nat.\ log.\ 10 = nat.\ log.\ (8+2)$$

$$= nat.\ log.\ 8 + 2\left[\frac{2}{16+2} + \tfrac{1}{3}\left(\frac{2}{16+2}\right)^3 + \cdots\right]$$

$$= 2{,}0794415 + 0{,}2231436 = 2{,}302585.$$

We can also put

$$nat.\ log.\ 2 = nat.\ log.\ 1 + 2\left[\frac{1}{4+1} + \tfrac{1}{3}\left(\frac{1}{4+1}\right)^3 + \cdots\right]$$

$$= 2\left(\tfrac{1}{5} + \tfrac{1}{3}\cdot\frac{1}{5^3} + \tfrac{1}{5}\cdot\frac{1}{5^5} + \cdots\right) = 0{,}693147, \text{ further,}$$

$$nat.\ log.\ 5 = nat.\ log.\ (4 + 1) = 2\ nat.\ log.\ 2 + 2\left(\tfrac{1}{9} + \tfrac{1}{3} \cdot \frac{1}{9^3} + \cdots\right),$$

and lastly, $nat.\ log.\ 10 = nat.\ log.\ 2 + nat.\ log.\ 5.$

(Comp. Art. 19.)

ART. 24. The *trigonometric* and *circular functions*, whose differentials are likewise to be found in the following, are also of practical importance.

The *function of sine* $y = sin.\ x$ gives, for $x = 0$, $y = 0$,

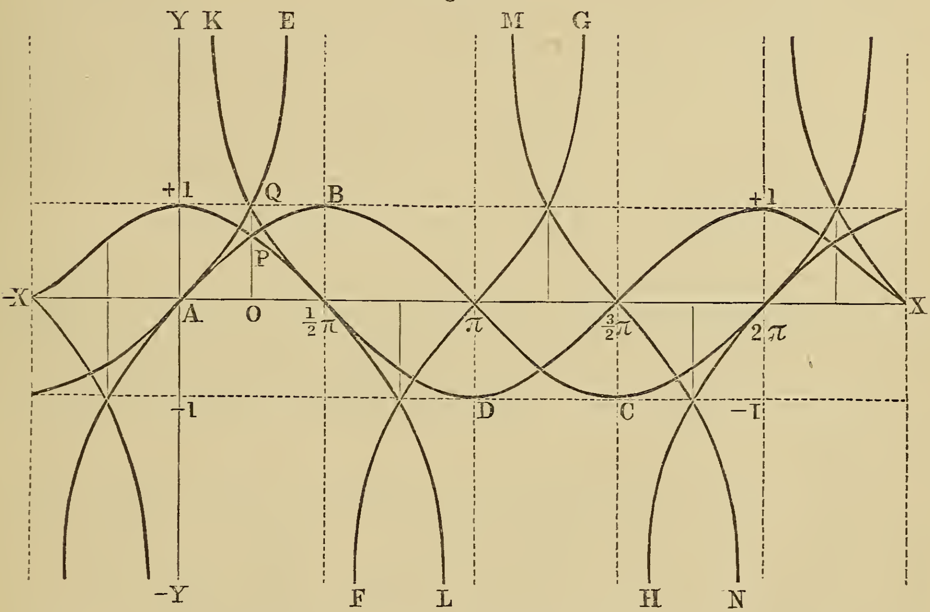
$$\text{for } x = \frac{\pi}{4} = \frac{3{,}1416}{4} = 0{,}7854\ldots, y = \sqrt{\tfrac{1}{2}} = 0{,}7071,$$

$$\text{for } x = \frac{\pi}{2}, y = 1, \text{ for } x = \pi, y = 0,$$

$$\text{for } x = \tfrac{3}{2}\pi, y = -1, \text{ for } x = 2\pi, y = 0, \text{ \&c.};$$

hence, if we take x as abscissa AO, and y as corresponding ordinate OP, we obtain the meandering curve $(APB\pi\ C\overline{2\pi})$, Fig. 33, which may be extended indefinitely on both sides of A.

Fig. 33.



The *function of cosine* $y = cos.\ x$ gives, for $x = 0$, $y = 1$, for $x = \frac{\pi}{4}$, $y = \sqrt{\frac{1}{2}}$, for $x = \frac{\pi}{2}$, $y = 0$, for $x = \pi$, $y = -1$, for $x = \frac{3}{2}\pi$, $y = 0$, for $x = 2\pi$, $y = 1$, &c.; therefore, precisely the same meandering line $\left(+1\, P\, \frac{\pi}{2}\, D\, \frac{3\pi}{2} + 1\right)$ which corresponds to the function of sine, corresponds also to the function of cosine; but it is, on the axis of abscissas, by $\frac{1}{2}\pi = 1{,}5708\ldots$ further before or behind the curve of the sine.

The curves, however, which correspond to the *tangential and cotangential functions* $y = tang.\ x$ and $y = cotang.\ x$, are of entirely

different form. If, in $y = tang.\ x$, we put $x = 0, \frac{1}{4}\pi, \frac{1}{2}\pi$, we obtain $y = 0, 1, \infty$; and hence, a curve $(A\,Q\,E)$ which approaches nearer and nearer to a line running parallel with the axis of ordinates $A\,Y$ and passing through the point $\left(\frac{\pi}{2}\right)$ of the axis of abscissas $A X$, but which never reaches it. If, further, we take $x = \frac{\pi}{2}, \pi, \frac{3}{2}\pi$, we obtain $y = -\infty, 0, +\infty$, and hence a curve $(F \pi\, G)$ which approaches the parallel lines running through $\left(\frac{\pi}{2}\right)$ and $(\frac{3}{2}\pi)$, *ad infinitum*, or which has, in other words, these parallels as *asymptotes*. (Vid. Art. 11.)

In assuming further values of x, the same values of y are repeated, whence also, the function $y = tang.\ x$ will be corresponded to by curves (as $F \pi\, G$) which are distant from one another by $\pi = 3{,}1416$ in the direction of the axis of abscissas.

The function $y = cotang.\ x$, on the other hand, gives for $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \pi$, $y = \infty, 1, 0, -\infty$; hence, to this corresponds a curve $\left(K\,Q\frac{\pi}{2}L\right)$ which differs from the tangential curve only in position. It is also easy to infer that innumerable other branches of curves, as, for example, $\left(M\frac{3\pi}{2}N,\right)$ &c., belong to this function.

Whilst both the curve for *sine* and that for *cosine* form a continuous whole, the *tangential* and *cotangential* curves consist of separate branches, inasmuch as their ordinates for certain values of x, pass from positive into negative infinity, whereby the curve loses its continuity.

ART. 25. The differentials of the *trigonometric lines or functions* are given in Fig. 34, in which we have
$C A = C P = C\,Q = 1$, $arc\ A\,P = x$, $arc\ P\,Q = \partial x$, further,
$P M = sin.\ x$, $C M = cos.\ x$, $A\,S = tang.\ x$, lastly,
$O\,Q = N\,Q - M P = sin.\ (x + \partial x) - sin.\ x = \partial\ sin.\ x$,
$O P = -(CN - CM) = - cos.\ (x + \partial x) - cos.\ x = - \partial\ cos.\ x$, and
$S\,T = A\,T - A\,S = tang.\ (x + \partial x) - tang.\ x = \partial\ tang.\ x$.

As the elementary arc $P Q$ stands at right angles to the radius CP, and the angle $P\,C\,A$ between the two lines $C\,P$ and CA is equivalent to the angle $P\,Q\,O$ between their perpendiculars $P\,Q$ and $O\,Q$, the triangles $C P M$ and $Q P O$ are *similar*, and we have

$$\frac{O\,Q}{P\,Q} = \frac{CM}{CP}, \text{ i. e. } \frac{\partial\ sin.\ x}{\partial x} = \frac{cos.\ x}{1}, \text{ hence:}$$

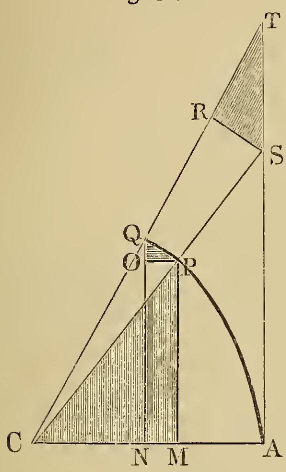
I. $\partial\,(sin.\,x) = cos.\,x\,.\,\partial x$; likewise also:

$$\frac{OP}{PQ} = \frac{PM}{CP},\ \text{i. e.}\ \frac{-\partial\,cos.\,x}{\partial x} = \frac{sin.\,x}{1},\ \text{hence:}$$

II. $\partial\,(cos.\,x) = -\,sin\,x\,\partial x$.

We see from this, that slight mistakes in the arc or angle have the more effect upon the sine, the greater $cos.\,x$ is, i. e. the smaller the arc or angle; but that they change the cosine the more, the greater $sin.\,x$ is, i. e. the more the arc approaches to $\frac{\pi}{2}$; and that, lastly, the differential of the cosine has a different sign from that of the arc, so that, as already known, an increase of x gives a decrease of $cos.\,x$, and inversely, a diminution of x, gives an increase of $cos.\,x$.

Fig. 34.



If SR be drawn at right angles to CT, there results a triangle SRT, which, on account of the equality of the angles RTS and CQN or CPM, is similar to the triangle CPM, and hence, we have:

$$\frac{ST}{SR} = \frac{CP}{CM},\ \text{i. e.}\ \frac{\partial\,tang.\,x}{SR} = \frac{1}{cos.\,x}.$$

But we have also: $\frac{SR}{CS} = \frac{PQ}{CP}$, i. e. $SR = \frac{CS\,.\,\partial x}{1}$ and

$$CS = secant.\,x = \frac{1}{cos.\,x},\ \text{hence}\ SR = \frac{\partial x}{cos.\,x}\ \text{and}$$

III. $\partial\,(tang.\,x) = \frac{\partial x}{(cos.\,x)^2}$.

If, for x, we substitute $\frac{\pi}{2} - x$, therefore, for ∂x, $\partial\left(\frac{\pi}{2} - x\right) = -\,\partial x$, we obtain:

$$\partial\,tang.\left(\frac{\pi}{2} - x\right) = -\,\frac{\partial x}{\left[cos.\left(\frac{\pi}{2} - x\right)\right]^2},\ \text{i. e.}$$

IV. $\partial\,(cotang\,x) = -\,\frac{\partial x}{(sin.\,x)^2}$.

By inversion, these formulae give, for the differential of the arc:

$$\partial x = \frac{\partial\,sin.\,x}{cos.\,x} = -\,\frac{\partial\,cos.\,x}{sin.\,x} = (cos.\,x)^2\,\partial\,tang.\,x$$

$$= -\,(sin.\,x)^2\,\partial\,cotang.\,x,\ \text{or:}$$

$$\partial x = \frac{\partial\,sin.\,x}{\sqrt{1 - (sin.\,x)^2}} = \frac{\partial\,tang.\,x}{1 + (tang.\,x)^2},$$

as also, $\partial x = -\dfrac{\partial\, cos.\, x}{\sqrt{1-(cos.\, x)^2}} = -\dfrac{\partial\, cotang.\, x}{1+(cotang\, x)^2}.$

If now *sin. x* be designated by y, and x by *arc.* (*sin.* $= y$), we obtain:

V. $\partial\, arc.\, (sin. = y) = \dfrac{\partial y}{\sqrt{1-y^2}};$

and in the same manner we find:

VI. $\partial\, arc.\, (cos. = y) = -\dfrac{\partial y}{\sqrt{1-y^2}}$, as also:

VII. $\partial\, arc.\, (tang. = y) = \dfrac{\partial y}{1+y^2}$, and

VIII. $\partial\, arc.\, (cotang. = y) = -\dfrac{\partial y}{1+y^2}.$

ART. 26. The last differential formulae give, by inversion, the following integral formulae:

I. $\int cos.\, x\, \partial\, x = sin.\, x,$

II. $\int sin.\, x\, \partial\, x = -\, cos.\, x,$

III. $\displaystyle\int \frac{\partial x}{cos.\, x^2} = tang.\, x,$

IV. $\displaystyle\int \frac{\partial x}{sin.\, x^2} = -\, cotang\ \ x$, further:

V. $\displaystyle\int \frac{\partial x}{\sqrt{1-x^2}} = arc.\, (sin. = x) = -\, arc.\, (cos. = x),$

VI. $\displaystyle\int \frac{\partial x}{1+x^2} = arc.\, (tang. = x) = -\, arc.\, (cotang. = x),$

and the following may also be easily found.

We have $\partial\, (nat.\, log.\, sin.\, x) = \dfrac{\partial\, sin.\, x}{sin.\, x} = \dfrac{cos.\, x\ .\ \partial\, x}{sin.\, x} = cotang.$ $x\ .\ \partial x$, consequently:

VII. $\int cotg.\, x \partial x = nat.\, log.\, sin.\, x$, likewise:

VIII. $\int tang.\, x \partial x = -\, nat.\, log.\, cos.\, x$, further:

$$\partial\, (nat.\, log.\, tang.\, x) = \frac{\partial\, tang.\, x}{tang.\, x} = \frac{\partial x}{cos.\, x^2\, tang\, x}$$

$$= \frac{\partial x}{sin.\, x\, cos.\, x} = \frac{\partial\, (2\, x)}{sin.\, 2\, x}, \text{ hence:}$$

$$\partial\, (nat.\, log.\, tang.\, \tfrac{1}{2}\, x) = \frac{\partial x}{sin.\, x}, \text{ and}$$

IX. $\displaystyle\int \frac{\partial x}{sin.\, x} = nat.\, log.\, tang.\, \frac{x}{2}$, likewise:

X. $\displaystyle\int \frac{\partial x}{cos.\, x} = nat.\, log.\, tang.\, \left(\frac{\pi}{4} + \frac{x}{2}\right)$

$$= nat.\, log.\, cotg.\, \left(\frac{\pi}{4} - \frac{x}{2}\right).$$

If, further, we put $\frac{1}{1-x^2} = \frac{a}{1+x} + \frac{b}{1-x} = \frac{a(1-x)+b(1+x)}{(1+x)(1-x)}$, there follows $1 = a\ (1-x) + b\ (1+x)$. If we take $1+x=0$, therefore, $x = -1$, we obtain $1 = a\ (1+1)$, as also $a = \frac{1}{2}$; and if we put $1 - x = 0$, therefore, $x = 1$, there results $1 = 2b$, hence:

$$b = \tfrac{1}{2} \text{ and } \frac{1}{1-x^2} = \frac{\frac{1}{2}}{1+x} + \frac{\frac{1}{2}}{1-x},$$

but lastly:

$$\int \frac{\partial x}{1-x^2} = \tfrac{1}{2}\int \frac{\partial x}{1+x} + \tfrac{1}{2}\int \frac{\partial x}{1-x}$$

$$= \tfrac{1}{2}\ \textit{nat. log.}\ (1+x) - \tfrac{1}{2}\ \textit{nat. log.}\ (1-x),$$

i. e.:

XI. $\int \frac{\partial x}{1-x^2} = \frac{1}{2}\ \textit{nat. log.} \left(\frac{1+x}{1-x}\right)$, and likewise:

XII. $\int \frac{\partial x}{x^2-1} = \frac{1}{2}\ \textit{nat. log.} \left(\frac{x-1}{x+1}\right)$.

If we put $\sqrt{1+x^2} = xy$, we obtain $1 + x^2 = x^2 y^2$, and $\partial x\ (1-y^2) = xy\,\partial y$, hence:

$$\frac{\partial x}{\sqrt{1+x^2}} = \frac{\partial y}{1-y^2} = \tfrac{1}{2}\ \partial\ \textit{nat. log.} \left(\frac{1+y}{1-y}\right), \text{ from which}$$

there follows

XIII. $\int \frac{\partial x}{\sqrt{1+x^2}} = \textit{nat. log.}\ (x + \sqrt{1+x^2})$, as also,

XIV. $\int \frac{\partial x}{\sqrt{x^2-1}} = \textit{nat. log.}\ (x + \sqrt{x^2-1})$.

Art. 27. To find *arc.* $(\textit{tang.} = x) = \int \frac{\partial x}{1+x^2}$, it is only necessary to develop $\frac{1}{1+x^2}$ into a series, by means of division, and then integrate each member. Thus we obtain

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots, \text{ and}$$

$$\int \frac{\partial x}{1+x^2} = \int \partial x - \int x^2 \partial x + \int x^4 \partial x - \int x^6 \partial x + \cdots,$$

consequently:

I. $\textit{arc.}\ (\textit{tang.} = x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$, for example:

$\frac{\pi}{4} = \textit{arc.}\ (\textit{tang.} = 1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$,

therefore, the semicircle $\pi = 4\ (1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots)$, or,

$\frac{\pi}{6} = \textit{arc.}\ (\textit{tang.} = \sqrt{\tfrac{1}{3}}) = \sqrt{\tfrac{1}{3}}\ [1 - \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{5}\ (\frac{1}{3})^2 - \frac{1}{7}\ (\frac{1}{3})^3 + \cdots]$,

consequently, $\pi = 6\sqrt{\tfrac{1}{3}}\ (1 - \frac{1}{9} + \frac{1}{45} - \frac{1}{189} + \cdots) = 3{,}1415926 \cdots$

In like manner we obtain

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} = 1 + \tfrac{1}{2}x^2 + \tfrac{3}{8}x^4 + \tfrac{5}{16}x^6 + \cdots$$

$$\int \frac{\partial x}{\sqrt{1-x^2}} = \int \partial x + \tfrac{1}{2}\int x^2 \partial x + \tfrac{3}{8}\int x^4 \partial x + \tfrac{5}{16}\int x^6 \partial x + \cdots,$$

i. e.:

$$\text{II. } arc.\,(sin. = x) = x + \frac{1\,x^3}{2.3} + \frac{1.3\,x^5}{2.4.5} + \frac{1.3.5\,x^7}{2.4.6.7} + \cdots,$$

for example:

$$\frac{\pi}{6} = arc.\,(sin. = \tfrac{1}{2}) = \tfrac{1}{2}\,(1 + \tfrac{1}{24} + \tfrac{3}{640} + \tfrac{5}{7168} + \cdots),$$

therefore:

$$\pi = 3.\left\{\begin{matrix} 1{,}04167 \\ 0{,}00469 \\ 0{,}00070 \\ 0{,}00012 \end{matrix}\right\} = 3{,}1416\ \ldots$$

There follows, further, from successive differentiation, if we put

$$sin.\,x = A_0 + A_1\,x + A_2\,x^2 + A_3\,x^3 + A_4\,x^4 + \cdots:$$

$$\frac{\partial\,(sin.\,x)}{\partial x} = cos.\,x = A_1 + 2\,A_2\,x + 3\,A_3\,x^2 + 4\,A_4\,x^3 + \cdots$$

$$\frac{\partial\,(cos.\,x)}{\partial x} = -\,sin.\,x = 2\,A_2 + 2\,.\,3\,A_3\,x + 3\,.\,4\,A_4\,x^2 + \cdots$$

$$-\frac{\partial\,(sin.\,x)}{\partial x} = -\,cos.\,x = 2\,.\,3\,.\,A_3 + 2\,.\,3\,.\,4\,.\,A_4\,x + \cdots$$

$$-\frac{\partial\,(cos.\,x)}{\partial x} = sin.\,x = 2\,.\,3\,.\,4\,.\,A_4 + \cdots$$

But we have for $x = 0$: $sin.\,x = 0$, and $cos.\,x = 1$; hence, there follows from the first series: $A_0 = 0$; from the second: $A_1 = cos.\,0 = 1$; from the third: $A_2 = 0$; from the fourth: $A_3 = -\frac{1}{2\,.\,3}$; from the fifth: $A_4 = 0$, &c.; and if these values are substituted in the assumed series, there results the series of sine:

$$\text{III. } sin.\,x = \frac{x}{1} - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3.4.5.6.7} + \text{\&c.}$$

In like manner we have

$$\text{IV. } cos.\,x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \text{\&c., further,}$$

$$\text{V. } tang.\,x = x + \frac{x^3}{3} + \frac{2\,x^5}{3.5} + \frac{17\,x^7}{3.5.7.9} + \cdots,$$

$$\text{VI. } cotang.\,x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{3.5.7} - \frac{2\,x^5}{3.5.7.9} - \text{\&c.}$$

ART. 28. If the differential formula $\partial\,(uv) = u\,\partial v + v\,\partial u$ of Art. 8, be integrated, there results the expression $u\,v = \int u\,\partial v + \int v\,\partial u$, and the following integral, known under the name, *reductions' formula:*

$$\int v \partial u = uv - \int u \partial v, \text{ or,}$$
$$\int \varphi(x) \partial f(x) = \varphi(x) f(x) - \int f(x) \partial \varphi(x).$$

This rule is always applicable when the integral $\int v \partial u = \int \varphi(x) \partial f(x)$ is not, but the integral $\int u \partial v = \int f(x) \partial \varphi(x)$ is, known.

By means of the reductions' formula, the integral from the following differential:

$$\partial y = \sqrt{1 + x^2} . \partial x,$$

may, for example, be reduced to another known integral. We have to put

$$\varphi(x) = \sqrt{1 + x^2}, \text{ therefore, } \partial \varphi(x) = \frac{x \partial x}{\sqrt{1 + x^2}}, \text{ and}$$
$$f(x) = x, \text{ therefore, } \partial f(x) = \partial x:$$

consequently, we have

$$\int \sqrt{1 + x^2} \partial x = x \sqrt{1 + x^2} - \int \frac{x^2 \partial x}{\sqrt{1 + x^2}},$$

but:

$$\frac{x^2}{\sqrt{1+x^2}} = \frac{1 + x^2}{\sqrt{1+x^2}} - \frac{1}{\sqrt{1+x^2}} = \sqrt{1+x^2} - \frac{1}{\sqrt{1+x^2}},$$

hence, there follows

$$\int \sqrt{1+x^2} \partial x = x \sqrt{1+x^2} - \int \sqrt{1+x^2} \partial x + \int \frac{\partial x}{\sqrt{1+x^2}},$$

or:

$$2 \int \sqrt{1+x^2} \partial x = x \sqrt{1+x^2} + \int \frac{\partial x}{\sqrt{1+x^2}},$$

and consequently:

I. $$\int \sqrt{1+x^2} \partial x = \tfrac{1}{2} x \sqrt{1+x^2} + \tfrac{1}{2} \int \frac{\partial x}{\sqrt{1+x^2}}$$
$$= \tfrac{1}{2} [x \sqrt{1+x^2} + nl. (x + \sqrt{1+x^2})];$$

likewise:

II. $$\int \sqrt{1-x^2} \partial x = \tfrac{1}{2} x \sqrt{1-x^2} + \tfrac{1}{2} \int \frac{\partial x}{\sqrt{1-x^2}}$$
$$= \tfrac{1}{2} [x \sqrt{1-x^2} + arc. (sin. = x)],$$

and

III. $$\int \sqrt{x^2-1} \partial x = \tfrac{1}{2} x \sqrt{x^2-1} - \tfrac{1}{2} \int \frac{\partial x}{\sqrt{x^2-1}}$$
$$= \tfrac{1}{2} [x \sqrt{x^2-1} - nl. (x + \sqrt{x^2-1})].$$

We have also

$$\int (sin. x)^2 \partial x = \int sin. x \, sin. x \partial x = -\int sin. x \partial (cos. x) = - sin. x \, cos. x$$
$$+ \int cos. x \partial (sin. x) = - sin. x \, cos. x + \int (cos. x)^2 \partial x$$
$$= - sin. x \, cos. x + \int [1 - (sin. x)^2] \partial x,$$

hence, there follows

$$2 \int (sin. x)^2 \partial x = \int \partial x - sin. x \, cos. x, \text{ and}$$

IV. $\int (sin.\, x)^2\, \partial x = \frac{1}{2}(x - sin.\, x\, cos.\, x) = \frac{1}{2}(x - \frac{1}{2}\, sin.\, 2\, x).$

There is, likewise,

V. $\int (cos.\, x)^2\, \partial x = \frac{1}{2}(x + sin.\, x\, cos.\, x) = \frac{1}{2}(x + \frac{1}{2}\, sin.\, 2\, x).$

We have, further,

VI. $\int sin.\, x\, cos.\, x\, \partial x = \frac{1}{4}\int sin.\, 2\, x\, \partial\, (2\, x) = -\frac{1}{4}\, cos.\, 2\, x,$

VII. $\int (tang.\, x)^2\, \partial x = tang.\, x - x$, and

VIII. $\int (cotg.\, x)^2\, \partial x = -(cotg.\, x + x).$

Finally, there is,

IX. $\int x\, sin.\, x\, \partial x = -x\, cos.\, x + \int cos.\, x\, \partial x = -x\, cos.\, x + sin.\, x,$

X. $\int x e^x \partial x = \int x\, \partial\, (e^x) = x e^x - \int e^x\, \partial x = (x - 1)\, e^x,$

XI. $\int nat.\, log.\, x\, .\, \partial x = x\, nat.\, log.\, x - \int x \frac{\partial x}{x} = x\, (nat.\, log.\, x - 1),$

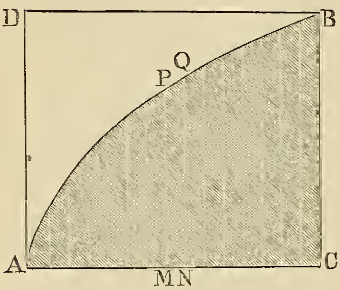
and

XII. $\int (x\, nat.\, log.\, x\, \partial x) = \frac{x^2}{2}\, nat.\, log.\, x - \int \frac{x^2}{2}\, \frac{\partial x}{x}$

$$= (nat.\, log.\, x - \tfrac{1}{2})\, \frac{x^2}{2}.$$

Art. 29. If it be required to *quadrate* a curve APB, Fig. 35, i. e., to determine or express the area of the surface ABC, which is bounded by the curve APB and its co-ordinates AC and BC, by a function of the abscissa of this curve, let us imagine this surface to be distributed by an infinite number of ordinates MP, NQ, &c., into laminated elements, as $MNQP$, of the constant breadth $MN = \partial x$ and the variable length $MP = y$. Since we may now put the area of such an element of surface,

Fig. 35.



$$\partial F = \left(\frac{MP + NQ}{2}\right) .\, MN = (y + \tfrac{1}{2}\, \partial y)\, \partial x = y\, \partial x,$$

the area of the entire surface F may be found by integrating the differential $y\, \partial x$, thus putting

$$F = \int y\, \partial x.$$

For example, for a *parabola* with the parameter p, we have $y^2 = px$, and hence the surface of the same:

$$F = \int \sqrt{px}\, \partial x = \sqrt{p} \int x^{\frac{1}{2}}\, \partial x = \frac{\sqrt{p}\, .\, x^{\frac{3}{2}}}{\frac{3}{2}} = \tfrac{2}{3}\, x \sqrt{px} = \tfrac{2}{3}\, xy.$$

The parabolic surface ABC is, therefore, two-thirds of the rectangle $ACBD$ which encloses it.

This formula is also applicable to *oblique angled* co-ordinates intersecting each other at an angle $XAY = \alpha$; for instance, to the surface ABC, Fig. 36, if, instead of $BC = y$, the normal distance $BN = y \sin. \alpha$ be substituted; we have here, therefore,

$$F = \sin. \alpha \int y \partial x.$$

For the parabolic surface, for example, when the axis of abscissas AX forms a diameter, and the axis of ordinates AY, a tangent, of the parabola, and therefore, $y^2 = p_1 x = \frac{px}{\sin \alpha^2}$, there results

$$F = \tfrac{2}{3}\, xy \sin. \alpha,$$

i. e.:

surface $ABC = \frac{2}{3}$ parallelogram $ACBD$.

Fig. 36.

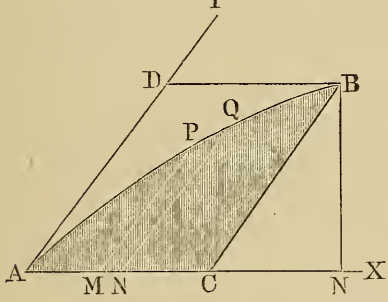
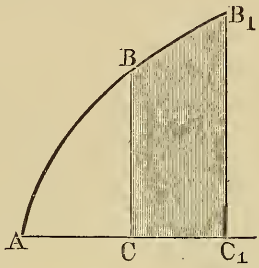


Fig. 37.



For a surface $BCC_1B_1 = F$, between the abscissas $AC_1 = c_1$ and $AC = c$, Fig. 37, there is, according to Art. 17,

$$F = \int_c^{c_1} y \partial x.$$

For $y = \frac{a^2}{x}$, there is, for example:

$$F = \int_c^{c_1} \frac{a^2 \partial x}{x} = a^2 (nat.\ log.\ c_1 - nat.\ log.\ c), \text{ i. e.:}$$

$$F = a^2\ nat.\ log. \left(\frac{c_1}{c}\right).$$

The curve PQ, Fig. 38, with which we have become familiar in Art. 3, corresponds to the equation $\frac{a^2}{x}$, and hence, if we have $AM = c$ and $AN = c_1$,

$$F = a^2\ nat.\ log. \left(\frac{c_1}{c}\right)$$

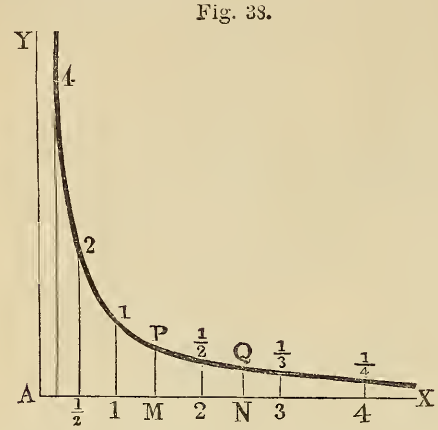
will give the area of $MNQP$. If, for the sake of simplicity, we assume $a = c = 1$, and $c_1 = x$, we obtain

$$F = nat.\ log.\ x,$$

and the areas $(1MP1)$, $(1NQ1)$, &c., are the natural logarithms of the abscissas AM, AN, &c. The curve itself is an *equilateral hyperbola*, in which the two semi-axes a and b are equal, consequently the angle of asymptotes $\alpha = 45^0$, and the straight lines AX

and $A\,Y$, to which the curve approaches nearer and nearer without reaching them, are the *asymptotes* of the same. On account of this connection between the abscissas and the areas, the *natural logarithms* are frequently called *hyperbolic logarithms.*

Fig. 38.

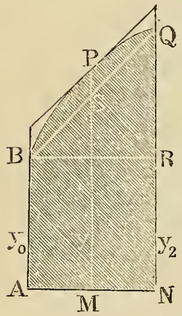


ART. 30. Every integral

$$\int y \partial x = \int \varphi(x) \partial x$$

may also be put equal to the area of a surface F, and if the integration cannot be performed by one of the known rules, the integral may be found, at least approximately, if the area of the corresponding surface be ascertained by the application of known geometrical rules.

Fig. 39.



For a surface $A\,B\,P\,Q\,N$, Fig. 39, which is determined by the base $A\,N = x$, and the three equi-distant ordinates $A\,B = y_0$, $M\,P = y_1$, and $N\,Q = y_2$, we have the trapezoidal portion

$$A\,B\,Q\,N = F_1 = (y_0 + y_2)\,\frac{x}{2},$$

and the segment formed portion $B\,P\,Q\,S\,B$, if $B\,P\,Q$ be regarded as a parabola,

$$F_2 = \tfrac{2}{3}\,PS \,.\, BR = \tfrac{2}{3}\,(MP - MS)\,.\,A\,N = \tfrac{2}{3}\left(y_1 - \frac{y_0 + y_2}{2}\right)x,$$

hence, the *entire surface:*

$$F = F_1 + F_2 = \left[\tfrac{1}{2}\,(y_0 + y_2) + \tfrac{2}{3}\left(y_1 - \frac{y_0 + y_2}{2}\right)\right]x$$

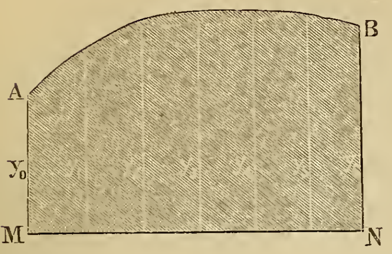
$$= \left[\tfrac{1}{6}\,(y_0 + y_2) + \tfrac{2}{3}\,y_1\right]x = (y_0 + 4\,y_1 + y_2)\,\frac{x}{6}.$$

If we introduce a *mean ordinate* y, and put $F = x\,y$, we obtain for the same:

$$y = \frac{y_0 + 4\,y_1 + y_2}{6}.$$

In order now to find from this the area of a surface $M\,A\,B\,N$, Fig. 40, which stands upon a given base $M\,N = x$, and is determined by an uneven number of ordinates $y_0, y_1, y_2, y_3 \,..\, y_n$, being distributed by these into an even number of strips of equal breadths, it is only necessary to repeat the application of the last rule. The breadth of one strip is $= \frac{x}{n}$, and from this the surface of the first couple of strips:

Fig. 40.



$$= \frac{y_0 + 4\, y_1 + y_2}{6} \cdot \frac{2x}{n},$$

of the second couple:

$$= \frac{y_2 + 4\, y_3 + y_4}{6} \cdot \frac{2x}{n},$$

of the third couple:

$$= \frac{y_4 + 4\, y_5 + y_6}{6} \cdot \frac{2x}{n}, \text{ \&c.};$$

therefore, the area of the first six strips, or first three couples, is, since we have here $n = 6$,

$$F = (y_0 + 4\, y_1 + 2\, y_2 + 4\, y_3 + 2\, y_4 + 4\, y_5 + y_6)\, \frac{x}{3.6}$$

$$= [y_0 + y_6 + 4\,(y_1 + y_3 + y_5) + 2\,(y_2 + y_4)]\, \frac{x}{18};$$

hence, it is easily inferred that the area of a surface distributed into four couples is

$$F = [y_0 + y_8 + 4\,(y_1 + y_3 + y_5 + y_7) + 2\,(y_2 + y_4 + y_6)]\, \frac{x}{3.8},$$

and that, generally, the area of a surface of n strips may be put

$$F = [y_0 + y_n + 4\,(y_1 + y_3 + \cdots + y_{n-1}) + 2\,(y_2 + y_4 + \cdots + y_{n-2})]\, \frac{x}{3n}.$$

The mean height of such a surface is:

$$y = \frac{y_0 + y_n + 4\,(y_1 + y_3 + \cdots + y_{n-1}) + 2\,(y_2 + y_4 + \cdots + y_{n-2})}{3\,n},$$

in which n must always be an even number.

This formula, known by the name of *Simpson's Rule*, is applicable in determining

an integral $\int_c^{c_1} y\,\partial x = \int_c^{c_1} \varphi\,(x)\,\partial x$, when we distribute $x = c_1 - c$ into an even number n of equal parts, calculate the ordinates

$$y_0 = \varphi\,(c),\, y_1 = \varphi\left(c + \frac{x}{n}\right),\, y_2 = \varphi\left(c + \frac{2x}{n}\right),$$

$$y_3 = \varphi\left(c + \frac{3x}{n}\right) \cdot\ \cdot, y_n = \varphi\,(x),$$

and introduce these values into the formula:

$$\int_c^{c_1} y\,\partial x = \int_c^{c_1} \varphi\,(x)\,\partial x$$

$$= [y_0 + y_n + 4\,(y_1 + y_3 + \cdots + y_{n-1}) + 2\,(y_2 + y_4 + \cdots + y_{n-2})]\, \frac{c_1 - c}{3\,n}.$$

For example, $\int_1^2 \frac{\partial x}{x}$ gives, since we have here $c_1 - c = 2 - 1 = 1$ and $y = \varphi\,(x) = \frac{1}{x}$, if we assume $n = 6$, and consequently $\frac{x}{n} = \frac{c_1 - c}{6} = \frac{1}{6}$:

$y_0 = \frac{1}{1} = 1{,}0000$, $y_1 = \frac{1}{\frac{7}{6}} = \frac{6}{7} = 0{,}8571$, $y_2 = \frac{1}{\frac{8}{6}} = \frac{3}{4} = 0{,}7500$, $y_3 = \frac{1}{\frac{9}{6}} = \frac{6}{9} = 0{,}6666$, $y_4 = \frac{1}{\frac{10}{6}} = 0{,}6000$, $y_5 = \frac{6}{11} = 0{,}5454$, and $y_6 = 0{,}5000$, hence:

$y_0 + y_6 = 1{,}5000$, $y_1 + y_3 + y_5 = 2{,}0692$, and $y_2 + y_4 = 1{,}3500$, and the integral sought:

$$\int_1^2 \frac{\partial x}{x} = (1{,}5000 + 4 \,.\, 2{,}0692 + 2 \,.\, 1{,}3500) \,.\, \tfrac{1}{18} = \frac{12{,}4768}{18} = 0{,}69315.$$

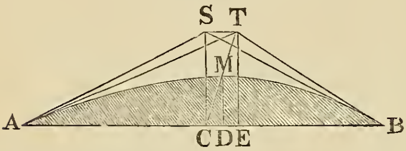
From Art. 22, III, we have

$$\int_1^2 \frac{\partial x}{x} = nat.\ log.\ 2 - nat.\ log.\ 1 = 0{,}693147,$$

therefore, the conformity is as desired.

ART. 31. In the sequel we shall give another rule, which may also be employed when the number n of strips is an uneven one. If a small segment AMB, Fig. 41, be treated as a parabolic segment, there follows, according to Art. 29, for the area of the same: $F = \frac{2}{3}\, AB \,.\, MD$,

Fig. 41.



or, if AT and BT are tangents to the ends A and B, and hence $CT = 2\,CM$: $F = \frac{2}{3} \,.\, \frac{AB \,.\, TE}{2} = \frac{2}{3}$ of the triangle $ATB = \frac{2}{3}$ of the equally high isosceles triangle ASB; and therefore also $= \frac{2}{3}\, AC \,.\, CS = \frac{2}{3}\, \overline{AC}^2 . tang.\ SAC$. The angle $SAC = SBC$ is $= TAC + TAS = TBC - TBS$; hence, if we put the small angles TAS and TBS equal to each other, we obtain for the same:

$$TAS = TBS = \frac{TBC - TAC}{2} \text{ and}$$

$$SAC = TAC + \frac{TBC - TAC}{2} = \frac{TAC + TBC}{2} = \frac{\delta + \varepsilon}{2},$$

if the tangential angles TAC and TBC are designated by δ and ε. Since we have, further, $AC = BC = \frac{1}{2} AB = \frac{1}{2}$ chord s, we have also,

$$F = \tfrac{1}{6} s^2\, tang. \left(\frac{\delta + \varepsilon}{2}\right).$$

This formula may now also be applied to the portion of surface $MABN$, Fig. 42, whose tangential angles $TAD = \alpha$ and $TBE = \beta$ are given. If, for example, we put the angle of chord $BAD = ABE = \sigma$, we obtain

$$TAB = \delta = TAD - BAD = \alpha - \sigma, \text{ and}$$
$$TBE = \varepsilon = ABE - TBE = \sigma - \beta, \text{ hence:}$$
$$\delta + \varepsilon = \alpha - \beta,$$

and the segment upon AB,

$$F = \tfrac{1}{6} s^2 \, tang. \left(\frac{\alpha - \beta}{2}\right).$$

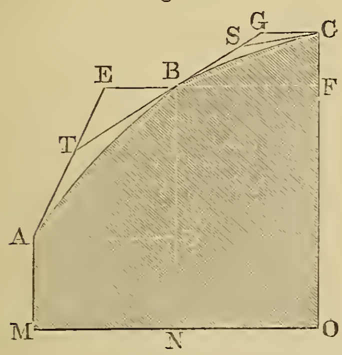
Now, as $\alpha - \beta$ is very small,

$$F = \frac{s^2}{12} \, tang. \, (\alpha - \beta) = \frac{s^2}{12} \left(\frac{tang. \, \alpha - tang. \, \beta}{1 + tang. \, \alpha \; tang. \, \beta}\right);$$

or, since α and β do not deviate largely from each other, and hence may be replaced in $tang. \, \alpha \; tang \, \beta$ by the mean value σ,

$$F = \tfrac{1}{12} s^2 \cdot \frac{tang. \, \alpha - tang. \, \beta}{1 + tang. \, \sigma^2} = \tfrac{1}{12} s^2 \cos. \sigma^2 \, (tang. \, \alpha - tang. \, \beta).$$

Fig. 42.



Therefore, by substituting for $s \cos. \sigma$ the base $MN = x$, we have

$$F = \frac{x^2}{12} \, (tang. \, \alpha - tang. \, \beta),$$

and hence the entire portion of surface $MABN$, if y_0 and y_1 represent its ordinates MA and NB:

$$F_1 = (y_0 + y_1) \frac{x}{2} + (tang. \, \alpha - tang. \, \beta) \frac{x^2}{12}.$$

If, with this portion of surface, there is an adjacent portion $NBCO$ having a like base $NO = x$, the ordinates BN and $CO = y_1$ and y_2, and the tangential angles $SBF = \beta$ and $SCG = \gamma$, the area of the same will be

$$F_2 = (y_1 + y_2) \frac{x}{2} + (tang. \, \beta - tang. \, \gamma) \frac{x^2}{12},$$

and hence the entire area, since here $- \, tang. \, \beta$ and $+ \, tang. \, \beta$ cancel each other:

$$F = F_1 + F_2 = (\tfrac{1}{2} y_0 + y_1 + \tfrac{1}{2} y_2) \, x + (tang. \, \alpha - tang. \, \gamma) \frac{x^2}{12}.$$

We have, likewise, for a surface consisting of three equally broad strips, if α represent the tangential angle of the initial, and δ that of the terminal, point:

$$F = (\tfrac{1}{2} y_0 + y_1 + y_2 + \tfrac{1}{2} y_3) \, x + (tang. \, \alpha - tang. \, \delta) \frac{x^2}{12},$$

and generally, for a portion of surface determined by the abscissas $\frac{x}{n}, \frac{2x}{n}, \frac{3x}{n} \cdots x$, the ordinates $y_0, y_1, y_2 \ldots y_n$, and the tangential angles α_0 and α_n of the terminal points:

$$F = (\tfrac{1}{2} y_0 + y_1 + y_2 + \cdots + y_{n-1} + \tfrac{1}{2} y_n) \frac{x}{n}$$
$$+ \tfrac{1}{12} (tang. \, \alpha - tang. \, \alpha_n) \left(\frac{x}{n}\right)^2.$$

An integral:

$$\int_c^{c_1} y\,\partial x = \int_c^{c_1} \varphi(x)\,\partial x$$

$$= (\tfrac{1}{2} y_0 + y_1 + y_2 + \cdots + y_{n-1} + \tfrac{1}{2} y_n) \frac{x}{n}$$

$$+ \tfrac{1}{12} (tang.\ a - tang.\ a_n) \left(\frac{x}{n}\right)^2$$

is found, accordingly, if we put $x = c_1 - c$, and calculate

$$y_0 = \varphi(c),\ y_1 = \varphi\left(c + \frac{x}{n}\right),\ y_2 = \varphi\left(c + \frac{2x}{n}\right),$$

$$y_3 = \varphi\left(c + \frac{3x}{n}\right) \ldots,\ y_n = \varphi\left(c + \frac{nx}{n}\right) = \varphi(c_1),$$

as also $tang.\ a = \dfrac{\partial y}{\partial x} = \psi(x) = \psi(c)$ and $tang.\ a_n = \psi(c_1)$, and introduce these values into this equation.

Thus if we assume $n = 6$, we obtain, for $\int_1^2 \dfrac{\partial x}{x}$, since there is here $x = c_1 - c = 2 - 1$ and $y = \varphi(x) = \dfrac{1}{x}$:

$$y_0 = \frac{1}{c} = 1,\ y_1 = \frac{1}{1 + \frac{1}{6}} = \tfrac{6}{7},\ y_2 = \tfrac{6}{8},\ y_3 = \tfrac{6}{9},$$

$$y_4 = \tfrac{6}{10},\ y_5 = \tfrac{6}{11}, \text{ and } y_6 = \tfrac{6}{12};$$

further, as there results $\dfrac{\partial y}{\partial x} = \dfrac{\partial (x^{-1})}{\partial x} = -\dfrac{1}{x^2}$:

$$tang.\ a = -\tfrac{1}{1} = -1 \text{ and } tang.\ \beta = -\left(\frac{1}{2}\right)^2 = -\tfrac{1}{4},$$

and hence there follows

$$\int_1^2 \frac{\partial x}{x} = (\tfrac{1}{2} + \tfrac{6}{7} + \tfrac{6}{8} + \tfrac{6}{9} + \tfrac{6}{10} + \tfrac{6}{11} + \tfrac{1}{4}) \cdot \tfrac{1}{6} + (-1 + \tfrac{1}{4}) \cdot \tfrac{1}{12} \cdot \tfrac{1}{36}$$

$$= \frac{4{,}1692}{6} - \tfrac{3}{4} \cdot \tfrac{1}{12} \cdot \tfrac{1}{36} = 0{,}69487 - 0{,}00173 = 0{,}69314.$$

(Comp. the example of the foregoing article.)

ART. 32. In order to *rectify* a curve, or, from its equation $y = f(x)$, between the co-ordinates $AM = x$ and $MP = y$, Fig. 43, to deduce an equation between the arc $AP = s$ and the one or the other of the two co-ordinates, we must first determine the differential of the arc AP, and then seek the integral. If x increase by $MN = PR = \partial x$, y will increase by $RQ = \partial y$, and s, by the element $PQ = \partial s$; and we have, in accordance with the *Pythagorean* theorem,

$$\overline{PQ}^2 = \overline{PR}^2 + \overline{QR}^2, \text{ i. e.: } \partial s^2 = \partial x^2 + \partial y^2;$$

therefore,

$$\partial s = \sqrt{\partial x^2 + \partial y^2}, \text{ and consequently, the arc itself:}$$

$$s = \int \sqrt{\partial x^2 + \partial y^2}.$$

For *Neil's parabola*, for example, (vid. Art. 9, Fig. 17,) the equation of which is $ay^2 = x^3$, we have $2 ay \partial y = 3 x^2 \partial x$; hence:

$$\partial y = \frac{3\, x^2 \partial x}{2\, ay} \text{ and } \partial y^2 = \frac{9\, x^4\, \partial x^2}{4\, a^2\, y^2} = \frac{9\, x \partial x^2}{4\, a},$$

according to which,

$$\partial s^2 = \left(1 + \frac{9x}{4a}\right) \partial x^2, \text{ and}$$

$$s = \int \sqrt{1 + \frac{9x}{4a}}\, \partial x = \frac{4a}{9} \int \left(1 + \frac{9x}{4a}\right)^{\frac{1}{2}} \partial \left(\frac{9x}{4a}\right)$$

$$= \frac{4a}{9} \int u^{\frac{1}{2}}\, \partial u = \frac{4a}{9}\, \tfrac{2}{3}\, u^{\frac{3}{2}} = \tfrac{8}{27}\, a \sqrt{\left(1 + \frac{9x}{4a}\right)^3}.$$

In order to find the constant which is here necessary, we will allow s to begin simultaneously with x and y. We thus obtain

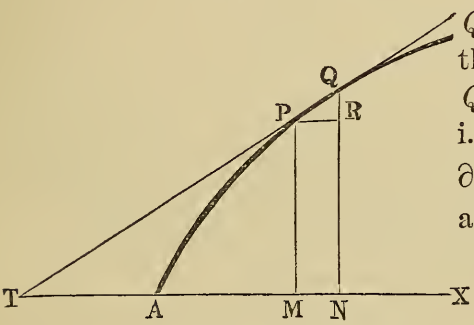
$$0 = \tfrac{8}{27}\, a \sqrt{1^3} + con., \text{ therefore } con. = -\tfrac{8}{27}\, a, \text{ and}$$

$$s = \tfrac{8}{27}\, a \left[\sqrt{\left(1 + \frac{9x}{4a}\right)^3} - 1\right],$$

for example, for the portion $A P_1$, having the abscissa $x = a$:

$$s = \tfrac{8}{27}\, a \left[\sqrt{(\tfrac{13}{4})^3} - 1\right] = 1{,}736\, a.$$

Fig. 43.



If, further, the tangential angle $QPR = PTM = \alpha$ be introduced, there results also, $QR = PQ\, .\, sin.\, QPR$ and $PR = PQ\, cos.\, QPR$, i. e.:

$\partial y = \partial s\, sin.\, \alpha$ and $\partial x = \partial s\, cos.\, \alpha$, and therefore, not only

$$tang.\, \alpha = \frac{\partial y}{\partial x} \text{ (Art. 6), but also}$$

$$sin.\, \alpha = \frac{\partial y}{\partial s} \text{ and } cos.\, \alpha = \frac{\partial x}{\partial s}; \text{ and, further,}$$

$$s = \int \sqrt{1 + tang.\, \alpha^2}\, .\, \partial x = \int \frac{\partial y}{sin.\, \alpha} = \int \frac{\partial x}{cos.\, \alpha}.$$

If, now, the equation between two of the magnitudes x, y, s, and α be given, we can then also find equations between two others of these magnitudes. If we have, for instance, $cos.\, \alpha = \frac{s}{\sqrt{c^2 + s^2}}$, there is also:

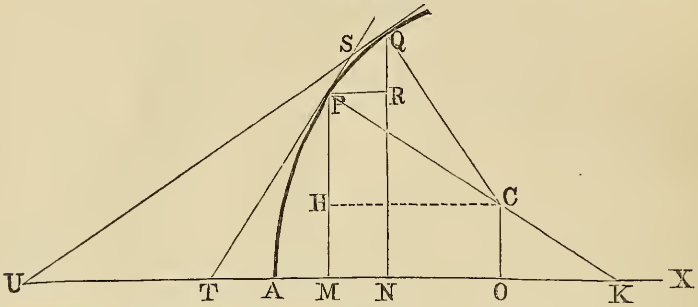
$$\partial x = \partial s\, cos.\, \alpha = \frac{s \partial s}{\sqrt{c^2 + s^2}} \text{ and}$$

$$x = \int \frac{s \partial s}{\sqrt{c^2 + s^2}} = \tfrac{1}{2} \int \frac{2 s \partial s}{\sqrt{c^2 + s^2}} = \tfrac{1}{2} \int \frac{\partial u}{\sqrt{u}} = \tfrac{1}{2} \int u^{-\frac{1}{2}} \partial u = u^{\frac{1}{2}} \partial u$$

$= \sqrt{c^2 + s^2} + con.$; and if x and s are, at the same time, zero: $x = \sqrt{c^2 + s^2} - c$.

ART. 33. A straight line at right angles to the tangent $P\,T$, Fig. 44, is also normal to the point of contact P of the curve; be-

Fig. 44.



cause the tangent indicates the direction of this point. The portion $P\,K$ of this line between the point of contact P and the axis of abscissas, is called simply, the *normal*, and the projection $M\,K$ of the same on the axis of abscissas, the *subnormal*. We have for the latter, since the angle $M\,P\,K$ is equal to the tangential angle $P\,T\,M = \alpha$,

$$MK = MP \;.\; tang.\; \alpha, \text{ i. e.:}$$

$$\text{the } subnormal = y\; tang.\; \alpha = y\,\frac{\partial y}{\partial x}.$$

As, for the system of curves $y = x^m$, there is $tang.\; \alpha = m\,x^{m-1}$, there follows here the *subnormal* $= m\,x^m \,.\, x^{m-1} = m\,x^{2m-1} = \frac{m\,y^2}{x}$; and for the *common parabola*, whose equation is $y^2 = p\,x$, we have

$$\text{the } subnormal = y\,\frac{p}{2y} = \frac{p}{2};$$

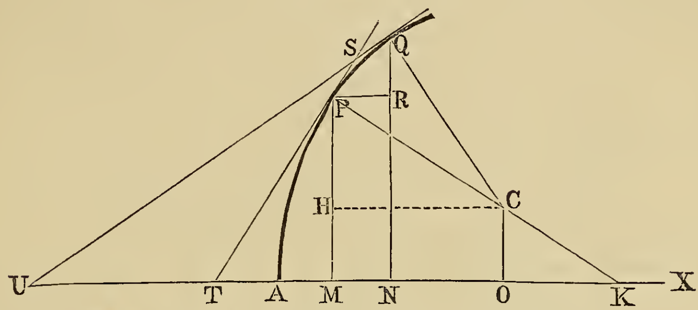
therefore *constant*.

If, further, another normal $Q\,C$ be drawn at a second point Q, infinitely near the point P, there results, in the point of intersection of the two lines, the centre C of a circle to be described through the two points of contact P and Q, or the *circle of gyration*; and the portions $C\,P$ and $C\,Q$ of the normals are the radii of this circle, or the *radii of gyration*. Of all the circles which may be drawn through P and Q, this is the one that most nearly coincides with the circular element $P\,Q$, and we may therefore assume that its arc $P\,Q$ coincides with that element.

If we designate the radius of gyration $CP = C\,Q$ by r, the circular arc $A\,P$ by s, therefore its element $P\,Q$ by ∂s, and the tangential angle or arc of $P\,T\,M$ by α, therefore its element $S\,U\,M$—

$S\,T\,M$, i. e. $-\,U\,S\,T = -\,P\,C\,Q$, by $\partial\alpha$, we have simply, since there is $P\,Q = C\,P$. arc of the angle $P\,C\,Q$: $\partial s = -\,r\,\partial\alpha$; and consequently, the radius of gyration: $r = -\dfrac{\partial s}{\partial \alpha}$.

Fig. 45.



Generally, α can only be determined by means of the equation of co-ordinates, since we put $tang.\ \alpha = \dfrac{\partial y}{\partial x}$.

But there is, further, $\partial\ tang.\ \alpha = \dfrac{\partial \alpha}{cos.\ \alpha^2}$ and $cos.\ \alpha = \dfrac{\partial x}{\partial s}$;

hence, we have $\partial\alpha = cos.\ \alpha^2 \,.\, \partial\ tang.\ \alpha = \dfrac{\partial x^2}{\partial s^2} \,.\, \partial\ tang.\ \alpha$, and

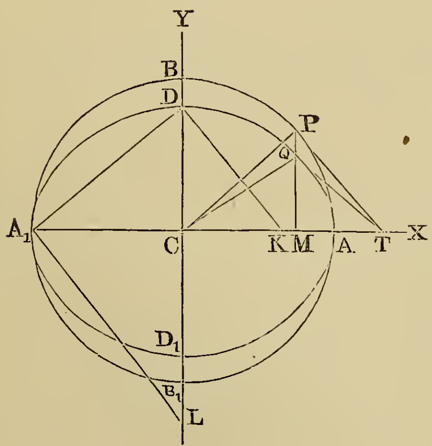
$$r = -\frac{\partial s}{cos.\ \alpha^2\,\partial\, tang.\, \alpha} = -\frac{\partial s^3}{\partial x^2\,\partial\, tang.\, \alpha}.$$

For a convex curve, we have $r = +\dfrac{\partial s}{\partial \alpha} = +\dfrac{\partial s^3}{\partial x^2\ \partial\ tang.\ \alpha}$; and for a point of inflection, $r = \infty$.

For the co-ordinates $A\,O = u$ and $O\,C = v$ of the centre C of gyration there is

$u = A\,M + H\,C = x + C\,P\ sin.\ C\,P\,H$, i. e. $u = x + r\ sin.\ \alpha$, and
$v = O\,C = M\,P - H\,P = y - C\,P\ cos.\ C\,P\,H$, i. e. $v = y - r\ cos.\ \alpha$.

Fig. 46.



The continuous succession of the centres of gyration gives a curve which is called the *evolute* of $A\,P$, and whose course is determined by the co-ordinates u and v.

If the *ellipse* $A\,D\,A_1\,D_1$ Fig. 46, be brought into connection with a circle $A\,B\,A_1\,B_1$, its co-ordinates $C\,M = x$ and $M\,Q = y$ may be expressed by the angle $P\,C\,B = \varphi$ at the centre of the circle. We have, for example,

$$x = CP \sin. CPM = CP \sin. BCP = a \sin. \varphi \text{ and}$$

$$y = MQ = \frac{b}{a} MP = \frac{b}{a} CP \cos. CPM = b \cos. \varphi.$$

From this there results

$$\partial x = a \cos. \varphi \partial \varphi \text{ and } \partial y = - b \sin. \varphi \partial \varphi,$$

consequently, for the *tangential angle* $QTX = \alpha$ of the *ellipse:*

$$tang. \alpha = \frac{\partial y}{\partial x} = - \frac{b \sin. \varphi}{a \cos. \varphi} = - \frac{b}{a} tang. \varphi,$$

and, for its adjacent angle $QTC = \alpha_1 = 180^0 - \alpha$:

$$tang. \alpha_1 = \frac{b}{a} tang. \varphi \text{ and } cotg. \alpha_1 = \frac{a}{b} cotg. \varphi.$$

Accordingly, the *subtangent* of the *ellipse* is

$$MT = MQ \, cotg. MTQ$$

$$= y \, cotg. \alpha_1 = \frac{ay}{b} cotg. \varphi = y_1 \, cotg. \varphi,$$

if y_1 designate the ordinate MP of the circle. Since, in the latter, the tangent PT stands at right angles to the radius CP, there is also $PTM = PCB = \varphi$, and hence, the sub-tangent of the same, likewise: $MT = MP \, cotg. MTP = y_1 \, cotg. \varphi$. Therefore, the two points P and Q of the circle and of the ellipse which have the same abscissas, have one and the same subtangent MT.

We have, further, for the elliptical elementary arc:

$$\partial s^2 = \partial x^2 + \partial y^2 = (a^2 \cos. \varphi^2 + b^2 \sin. \varphi^2) \partial \varphi^2,$$

and the differential of *tang.* α:

$$\partial \, tang. \alpha = - \frac{b}{a} \partial \, tang. \varphi = - \frac{b}{a} \frac{\partial \varphi}{\cos. \varphi^2};$$

hence, there follows the *radius of curvature* of the ellipse:

$$r = - \frac{\partial s^3}{\partial x^2 \partial \, tang. \alpha} = \frac{(a^2 \cos. \varphi^2 + b^2 \sin. \varphi^2)^{\frac{3}{2}}}{a^2 \cos. \varphi^2 \cdot \frac{b}{a \cos. \varphi^2}}$$

$$= \frac{(a^2 \cos. \varphi^2 + b^2 \sin. \varphi^2)^{\frac{3}{2}}}{ab}.$$

For example, for $\varphi = 0$, therefore $\sin. \varphi = 0$ and $\cos. \varphi = 1$, there follows the *greatest* radius of curvature:

$$r_m = \frac{a^3}{ab} = \frac{a^2}{b},$$

and on the other hand, for $\varphi = 90^0$, therefore $\sin. \varphi = 1$ and $\cos. \varphi = 0$, the *smallest* radius of curvature:

$$r_n = \frac{b^3}{ab} = \frac{b^2}{a}.$$

The first value of r corresponds to the point D, and the last, to the point A; both are determined by the portions of axes CL and CK, which, from C, cut off at the extremities A_1 and D the perpendiculars erected upon the chord $A_1 D$.

Art. 34. Many functions which occur in practice may be composed of the principal functions treated of above:

$$y = x^m,\ y = e^x, \text{ and } y = sin.\ x,\ y = cos.\ x, \ \&c\ ,$$

and it is also easy, by aid of the foregoing, to find their properties with respect to the position of tangents, quadrature, radii of curvature, &c., as also, to construct the curves corresponding to them, as will be shown in the following example.

For the curve corresponding to the equation:

$$y = x^2 \left(1 - \frac{x}{3}\right) = x^2 - \tfrac{1}{3} x^3, \text{ we have}$$

$$\partial y = 2x\partial x - x^2 \partial x, \text{ consequently,}$$

$$tang.\ \alpha = 2x - x^2 = x\ (2 - x).$$

As this tangent is $= 0$ for $x = 0$ and $x = 2$, it has, also, at the points which appertain to these values of abscissas, the direction of the axis of abscissas. There is further:

$$\partial\ tang.\ \alpha = 2\partial x - 2x\partial x = 2\ (1 - x)\ \partial x,$$

according to which, we have

for $x = 0$, $\partial\ tang.\ \alpha = +\ 2\partial x$, and

for $x = 2$, $\partial\ tang.\ \alpha = -\ \partial x$;

and hence, the ordinate of the first point is a *minimum*, whilst that of the second is a *maximum*. If we put $\partial\ tang.\ \alpha = 0$, we obtain thereby, the co-ordinates, $x = 1$ and $y = \frac{2}{3}$, of the *point of inflection* at which the concave portion of the curve joins the convex portion.

We have, further, for the elementary curve ∂s:

$$\partial s^2 = \partial x^2 + \partial y^2 = \partial x^2 + x^2\ (2 - x)^2\ \partial x^2 = [1 + x^2\ (2 - x)^2]\ \partial x^2,$$

and hence the radius of curvature of the curve:

$$r = -\frac{\partial s^3}{\partial x^2 \partial\ tang.\ \alpha} = -\frac{[1 + x^2\ (2 - x)^2]^{\frac{3}{2}}}{2\ (1 - x)},$$

for example,

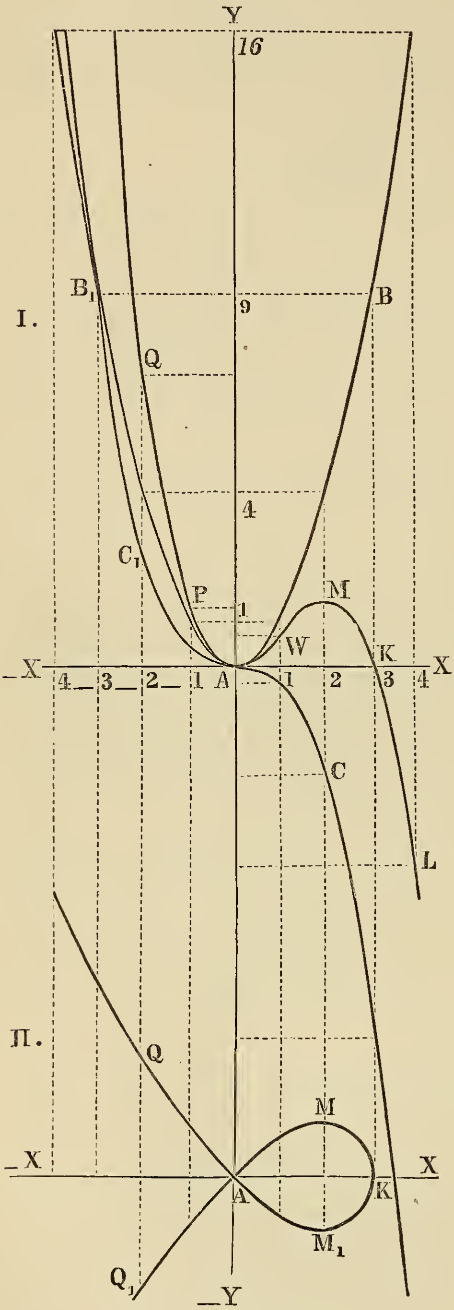
$$\text{for } x = 0,\ r = \frac{-1}{2} = -\tfrac{1}{2}, \text{ for } x = 1,\ r = -\frac{2^{\frac{3}{2}}}{0} = \infty,$$

$$\text{for } x = 2,\ r = \frac{-1}{-2} = +\tfrac{1}{2}, \text{ for } x = 3,\ r = \tfrac{1}{4} \, .\, 10^{\frac{3}{2}} = +\ 7{,}906.$$

The corresponding curve is shown in Fig. 47, in which A represents the origin of the co-ordinates, and $\overline{XX}$, $\overline{YY}$, the axes of the co-ordinates. To the first part, $y_1 = x^2$, of the equation, corresponds the parabola BAB_1, which, from A, passes symmetrically to both sides of the axis AY; and to the second part, $y_2 = -\frac{1}{3} x^3$, corresponds the curve CAC_1, which, on the right side of $\overline{YY}$ passes below the axis of abscissas $\overline{XX}$, whilst on the left, it passes above $\overline{XX}$, and withdraws farther and farther from the same the more it is removed from $\overline{YY}$. To determine, for a given axis of abscissas x, the cor-

responding point of the curve $y = x^2 - \frac{1}{3}x^3$, it is only necessary to add algebraically the ordinates of the first curves, which appertain to this abscissa. As we have, for example, for $x = 1$, $y_1 = 1$ and $y_2 = -\frac{1}{3}$, there follows the corresponding ordinate of the point W: $y = y_1 + y_2 = 1 - \frac{1}{3} = \frac{2}{3}$; further, since we have for $x = 2$, $y_1 = 4$ and $y_2 = -\frac{8}{3}$, there follows also the co-ordinate of the point M: $y = 4 - \frac{8}{3} = \frac{4}{3}$. Likewise, we obtain for $x = 3$, $y = y_1 + y_2 = 9 - 9 = 0$, for $x = 4$, $y = 16 - \frac{64}{3} = -\frac{16}{3}$, for $x = -1$, $y = 1 + \frac{1}{3} = \frac{4}{3}$, for $x = -2$, $y = 4 + \frac{8}{3} = \frac{20}{3}$, &c., and we perceive that the last curve has, from A to the right side, the course $AWMKL\ldots$, at the commencement of which it passes along above the abscissa $AK = 3$, but that after the point K it runs below $X\overline{X}$ indefinitely, whilst to the left of A, it ascends constantly, forming the indefinite branch $APQ\ldots$ From the above also, W is a point of inflection, and M a maximum point, of the curve. Whilst at A and M the curve has the direction of $X\overline{X}$, at W it ascends at an angle $\alpha = 45^0$, because we have for the same *tang.* $\alpha = x\,(2 - x) = 1$; but for the angle of inclination at K, there is *tang.* $\alpha = -3$, consequently, $\alpha = 71^0\,34'$, &c.

Fig. 47.



The *quadrature of the curve* is performed by the integral

$$F = \int y\,\partial x = \int (x^2 - \tfrac{1}{3}\,x^3)\,\partial x = \int x^2\,\partial x - \tfrac{1}{3}\int x^3\,\partial x$$

$$= \frac{x^3}{3} - \frac{x^4}{12} = \frac{x^3}{3}\left(1 - \frac{x}{4}\right).$$

Accordingly, there follows, for example, for the portion of surface

$AWMK$ above $AK = 3$, the area $F = \frac{3^3}{3}(1 - \frac{3}{4}) = \frac{9}{4}$, and, on the other hand, for the portion $\overline{3L4}$ below the portion $\overline{34}$ of the abscissa,

$$F_1 = \frac{4^3}{3}(1 - \tfrac{4}{4}) - \frac{3^3}{3}(1 - \tfrac{3}{4}) = 0 - \tfrac{9}{4} = - \tfrac{9}{4}.$$

Lastly, to find the length of a portion of a curve, as AWM, we put

$$s = \int \sqrt{1 + x^2 (2 - x)^2}\, \partial x = \int_c^{c_1} \varphi(x)\, \partial x,$$

and apply the method of integration discussed in Art. 30. We have here $c = 0$ and $c_1 = 2$; if we assume $n = 4$, there follows $\partial x = \frac{c_1 - c}{n} = \frac{2 - 0}{4} = \frac{1}{2}$, and if in the function $\varphi(x) = \sqrt{1 + x^2 (2 - x)^2}$, we substitute for x the values $0, \frac{1}{2}, 1, \frac{3}{2}, 2$, successively, there will result:

$$\varphi(0) = \sqrt{1} = 1, \varphi(\tfrac{1}{2}) = \sqrt{1 + \tfrac{9}{16}} = \tfrac{5}{4},$$
$$\varphi(1) = \sqrt{1 + 1} = \sqrt{2} = 1{,}414 \ldots$$
$$\varphi(\tfrac{3}{2}) = \sqrt{1 + \tfrac{9}{16}} = \tfrac{5}{4}, \text{ and } \varphi(2) = \sqrt{1} = 1,$$

and hence the length of the arc AWM:

$$s = \left(\varphi(0) + 4\varphi(\tfrac{1}{2}) + 2\varphi(1) + 4\varphi(\tfrac{3}{2}) + \varphi(2)\right) \frac{c_1 - c}{3 \cdot 4}$$
$$= (1 + 5 + 2{,}828 + 5 + 1) \cdot \tfrac{1}{6} = 2{,}471.$$

By means of the curve $y = x^2 \left(1 - \frac{x}{3}\right)$, the course of the curve $y = x\sqrt{1 - \frac{x}{3}}$ may now also be easily indicated; for if we extract the square root of the co-ordinate values of the first, there result the corresponding co-ordinates of the last. As the square roots of negative quantities are imaginary, this curve does not extend beyond K; and since every square root of positive quantities has two equally great and opposite values, the new curve (II.) consists of two symmetrical branches $QAMK$ and Q_1AM_1K, one on each side of the axis $X\overline{X}$.

Art. 35. When the quotient $y = \frac{\varphi(x)}{\psi(x)}$ of two functions $\varphi(x)$ and $\psi(x)$ assumes, for a certain value a of x, the indeterminate value $\frac{0}{0}$,—which is always the case when, as in $y = \frac{x^2 - a^2}{x - a}$, the numerator and denominator of a fraction have one factor $x - a$ in common —, the real value of the same may be arrived at by differentiating the numerator and denominator separately.

If x increase by the element ∂x, and y, correspondingly, by the element ∂y, there results:

$$y + \partial y = \frac{\varphi(x) + \partial\varphi(x)}{\psi(x) + \partial\psi(x)}.$$

But we have now for $x = a$:

$$\varphi(x) = 0 \text{ and } \psi(x) = 0;$$

hence, there is for this case:

$$y + \partial y = \frac{\partial\varphi(x)}{\partial\psi(x)},$$

or, since ∂y as infinitely small magnitude vanishes in comparison with y:

$$y = \frac{\varphi(x)}{\psi(x)} = \frac{\partial\varphi(x)}{\partial\psi(x)} = \frac{\varphi_1(x)}{\psi_1(x)},$$

in which $\varphi_1(x)$ and $\psi_1(x)$ designate the differential quotients of $\varphi(x)$ and $\psi(x)$.

If $y = \dfrac{\varphi_1(x)}{\psi_1(x)}$ is again $= \dfrac{0}{0}$, we can differentiate anew, and put

$$y = \frac{\partial\varphi_1(x)}{\partial\psi_1(x)} = \frac{\varphi_2(x)}{\psi_2(x)}, \text{ \&c.}$$

The undetermined expressions $y = \frac{\infty}{\infty}$, $0 \cdot \infty$, &c., may be treated in like manner, since we may put $\infty = \dfrac{1}{0}$, consequently $\frac{\infty}{\infty}$, and $0 \cdot \infty, = \dfrac{0}{0}$.

For example:

$y = \dfrac{3x^3 - 7x^2 - 8x + 20}{5x^3 - 21x^2 + 24x - 4}$ gives for $x = 2$, $\dfrac{0}{0}$; hence, it is also admissable to put

$$y = \frac{\partial(3x^3 - 7x^2 - 8x + 20)}{\partial(5x^3 - 21x^2 + 24x - 4)} = \frac{9x^2 - 14x - 8}{15x^2 - 42x + 24}.$$

For $x = 2$, however, y is still $= \dfrac{0}{0}$; hence, we put again:

$$y = \frac{\partial(9x^2 - 14x - 8)}{\partial(15x^2 - 42x + 24)} = \frac{18x - 14}{30x - 42} = \frac{9x - 7}{15x - 21} = \frac{11}{9}.$$

But the factor $x - 2$ is also in reality contained twice in the numerator and denominator of the given function. If both are divided by $x - 2$, there results

$$y = \frac{3x^2 - x - 10}{5x^2 - 11x + 2},$$

and if this division be repeated in the last value:

$$y = \frac{3x + 5}{5x - 1},$$

therefore, putting $x = 2$, there follows $y = \dfrac{11}{9}$.

Further, $y = \frac{a - \sqrt{a^2 - x}}{x}$ gives for $x = 0$, $\frac{0}{0}$.

But we have

$$\partial\,(a - \sqrt{a^2 - x}) = -\,\partial\,(a^2 - x)^{\frac{1}{2}} = \frac{\frac{1}{2}\,\partial x}{\sqrt{a^2 - x}},$$

hence, there follows for this case, $y = \frac{\frac{1}{2}}{\sqrt{a^2 - x}} = \frac{1}{2a}$.

If, further, in $y = \frac{nat.\,log.\,x}{\sqrt{1 - x}}$, we put $x = 1$, there follows $y = \frac{0}{0}$; but we have now,

$$\partial\; nat.\,log.\,x = \frac{\partial x}{x}, \text{ and } \partial \sqrt{1 - x} = -\frac{\partial x}{2\sqrt{1 - x}}; \text{ hence:}$$

$$y = -\frac{2\sqrt{1 - x}}{x} = \frac{2\,.\,0}{1} = 0.$$

Lastly,

$$y = \frac{1 - sin.\,x + cos.\,x}{-1 + sin.\,x + cos.\,x} \text{ gives for } x = \frac{\pi}{2}\ (90^0),$$

$$y = \frac{1 - 1 + 0}{-1 + 1 + 0} = \frac{0}{0};$$

hence, we must also put

$$y = \frac{\partial\,(1 - sin.\,x + cos.\,x)}{\partial\,(-1 + sin.\,x + cos.\,x)} = \frac{-\,cos.\,x - sin.\,x}{cos.\,x - sin.\,x}$$

$$= \frac{-0 - 1}{0 - 1} = 1.$$

ART. 36. If, for a function $y = \alpha u + \beta v$, a series of associated values of the variables u, v, and y, has been found by observation or measurement, we may also demand those *values* of the constants α and β which are as free as possible from incidental and irregular errors of observation and measurement, and which, therefore, express as accurately as possible the connection between the magnitudes u, v, and y, of which u and v may also signify known functions of one and the same variable, x. Of all the rules which are employed in determining those *values* of the constants which are supposed to be the most accurate, the *method of the least squares* has the most general and scientific foundation.

If

$$\begin{pmatrix} u_1, & v_1, & y_1 \\ u_2, & v_2, & y_2 \\ u_3, & v_3, & y_3 \\ . & . & . \\ . & . & . \\ u_n, & v_n, & y_n \end{pmatrix}$$

are the observed results corresponding to the function $y = \alpha u + \beta v$, we have, for the errors of observation and their squares, the following values:

$$\left\{\begin{array}{l} z_1 = y_1 - (\alpha u_1 + \beta v_1) \\ z_2 = y_2 - (\alpha u_2 + \beta v_2) \\ z_3 = y_3 - (\alpha u_3 + \beta v_3) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ z_n = y_n - (\alpha u_n + \beta v_n) \end{array}\right\}$$

and

$$\left\{\begin{array}{l} z_1^2 = y_1^2 - 2\alpha u_1 y_1 - 2\beta v_1 y_1 + \alpha^2 u_1^2 + 2\alpha\beta u_1 v_1 + \beta^2 v_1^2 \\ z_2^2 = y_2^2 - 2\alpha u_2 y_2 - 2\beta v_2 y_2 + \alpha^2 u_2^2 + 2\alpha\beta u_2 v_2 + \beta^2 v_2^2 \\ z_3^2 = y_3^2 - 2\alpha u_3 y_3 - 2\beta v_3 y_3 + \alpha^2 u_3^2 + 2\alpha\beta u_3 v_3 + \beta^2 v_3^2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ z_n^2 = y_n^2 - 2\alpha u_n y_n - 2\beta v_n y_n + \alpha^2 u_n^2 + 2\alpha\beta u_n v_n + \beta^2 v_n^2 \end{array}\right\}$$

and if, for abridgement, we employ the sign (Σ) of summation, in order to indicate a summation of homogeneous magnitudes, thus putting $y_1^2 + y_2^2 + y_3^2 + \cdots + y_n^2 = \Sigma(y^2)$, $v_1 y_1 + v_2 y_2 + v_3 y_3 + \cdots + v_n y_n = \Sigma(vy)$, &c., we obtain for the sum of the squares of errors:

$$\Sigma(z^2) = \Sigma(y^2) - 2\alpha\Sigma(uy) - 2\beta\Sigma(vy) + \alpha^2\Sigma(u^2) + 2\alpha\beta\Sigma(uv) + \beta^2\Sigma(v^2).$$

In this equation, only the constants α and β of the function $y = \alpha u + \beta v$, which are to be regarded as independent, are unknown; excepting, of course, the sum of the squares of errors $\Sigma(z^2)$, which must be treated as a dependent variable. The method of the least squares demands now that α as well as β be so chosen that the sum of the squares $\Sigma(z^2)$ may become a minimum; and hence, we must differentiate the function obtained for $\Sigma(z^2)$; first, with respect to α, and again, with respect to β, and put each of these differential quotients $= 0$. In this manner we arrive at the following equations of condition for α and β:

$$-\Sigma(uy) + \alpha\Sigma(u^2) + \beta\Sigma(uv) = 0,$$
$$-\Sigma(vy) + \beta\Sigma(v^2) + \alpha\Sigma(uv) = 0;$$

the solution of which leads to the following expressions:

$$\alpha = \frac{\Sigma(v^2)\,\Sigma(uy) - \Sigma(uv)\,\Sigma(vy)}{\Sigma(u^2)\,\Sigma(v^2) - \Sigma(uv)\,\Sigma(uv)} \text{ and}$$

$$\beta = \frac{\Sigma(u^2)\,\Sigma(vy) - \Sigma(uv)\,\Sigma(uy)}{\Sigma(u^2)\,\Sigma(v^2) - \Sigma(uv)\,\Sigma(uv)}.$$

As we have here $u = 1$, therefore $\Sigma(uv) = \Sigma(v)$, $\Sigma(uy) = \Sigma(y)$, and $\Sigma(u^2) = 1 + 1 + 1 + \cdots = n$, i. e. the number of the equations or observations, the above formulae pass into the following:

$$\alpha = \frac{\Sigma(v^2)\,\Sigma(y) - \Sigma(v)\,\Sigma(vy)}{n\,\Sigma(v^2) - \Sigma(v)\,\Sigma(v)},$$

$$\beta = \frac{n\,\Sigma(vy) - \Sigma(v)\,\Sigma(y)}{n\,\Sigma(v^2) - \Sigma(v)\,\Sigma(v)}.$$

For the still more simple function $y = \beta v$, where we have $\alpha = 0$, there results

$$\beta = \frac{\Sigma(vy)}{\Sigma(v^2)},$$

and lastly, for the simplest case, $y = \alpha$, where it is therefore requisite to find the most probable value of one single magnitude, we have

$$\alpha = \frac{\Sigma(y)}{n}$$

as *arithmetical mean* of all the values found by measurement or observation.

Example. — To become acquainted with the law of uniformly accelerated motion, i. e. with its initial velocity c and its measure of acceleration p, the spaces s_1, s_2, s_3, &c., corresponding to the different times t_1, t_2, t_3, &c., have been measured, and the following is the result.

Times	0	1	3	5	7	10 Sec.
Spaces	0	5	20	38	$58\frac{1}{2}$	101 ft.

If now $s = ct + \frac{pt^2}{2}$ be the law of this motion, we have to find the constants c and p. If, in the above formulae, we put $u = t$, $v = t^2$, $\alpha = c$, $\beta = \frac{p}{2}$, and $y = s$, there result the following formulae for the calculation of c and p:

$$c = \frac{\Sigma(t^4)\,\Sigma(st) - \Sigma(t^3)\,\Sigma(st^2)}{\Sigma(t^2)\,\Sigma(t^4) - \Sigma(t^3)\,\Sigma(t^3)} \text{ and}$$

$$\frac{p}{2} = \frac{\Sigma(t^2)\,\Sigma(st^2) - \Sigma(t^3)\,\Sigma(st)}{\Sigma(t^2)\,\Sigma(t^4) - \Sigma(t^3)\,\Sigma(t^3)},$$

whence, the following:

t	t^2	t^3	t^4	s	st	st^2
1	1	1	1	5	5	5
3	9	27	81	20	60	180
5	25	125	625	38	190	950
7	49	343	2401	58,5	409,5	2866,5
10	100	1000	10000	101	1010	10100
Amounts	184 $= \Sigma(t^2)$	1496 $= \Sigma(t^3)$	13108 $= \Sigma(t^4)$	222,5 $= \Sigma(s)$	1674,5 $= \Sigma(st)$	14101,5 $= \Sigma(st^2)$.

From the above there results

$$c = \frac{13108\ .\ 1674{,}5 - 1496\ .\ 14101{,}5}{184\ .\ 13108 - 1496\ .\ 1496} = \frac{85340}{17386} = 4{,}908 \text{ ft. and}$$

$$\tfrac{1}{2}p = \frac{184\ .\ 14101{,}5 - 1496\ .\ 1674{,}5}{184\ .\ 13108 - 1496\ .\ 1496} = \frac{89624}{173860} = 0{,}5155 \text{ ft.},$$

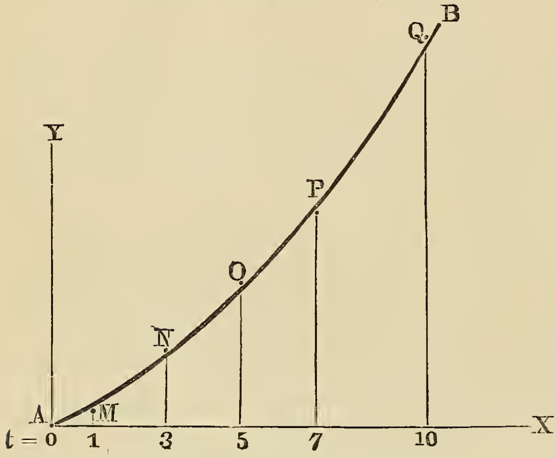
and hence, the following formula for the observed motion:

$$s = 4{,}908\ t + 0{,}5155\ .\ t^2.$$

According to this formula, we have

for the times . .	0	1	3	5	7	10 sec.,
the spaces . . .	0	5,43	19,36	37,43	59,62	100,63 ft.

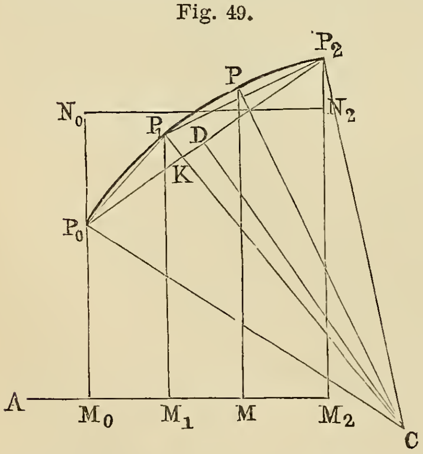
Fig. 48.



If the times (t) be taken as abscissas, and the observed, as well as the estimated, spaces (s) as ordinates, a curve AB, Fig. 48, may be drawn through the ends of the estimated co-ordinates, which will pass between the points M, N, O, P, Q determined by the observed co-ordinates, in such a manner that the sum of the squares of the deviation of the same from these points will be as small as possible on both sides.

ART. 37. If, in default of a formula for the constant progression of a magnitude y, or its dependence upon another magnitude x, it be necessary to determine a value of the magnitude y which corresponds to a given value of x, by means of the values of x and y as known by experience or taken from a table, we must make use of the *process of interpolation*, of which only the most important part is to be communicated here.

Fig. 49.



If the abscissas $AM_0 = x_0$, $AM_1 = x_1$, and $AM_2 = x_2$, Fig. 49, and the corresponding ordinates $M_0P_0 = y_0$, $M_1P_1 = y_1$, and $M_2P_2 = y_2$ are given, the ordinate $MP = y$ corresponding to a new abscissa $AM = x$, may be expressed by the formula $y = \alpha + \beta x + \gamma x^2$, provided the three points P_1, P_2, P_3 thus

determined, lie in a nearly straight line, or in a slightly curved arc. If the initial point of the co-ordinates be transferred from A to M_0, the universality does not suffer, but we obtain simply $y = a$ for $x = 0$, and consequently, the constant member $a = y_0$.

If now we introduce into the assumed equation, first, x_1 and y_1, and again, x_2 and y_2, we obtain the two following equations of condition:

$$y_1 - y_0 = \beta x_1 + \gamma x_1^2 \text{ and}$$
$$y_2 - y_0 = \beta x_2 + \gamma x_2^2, \text{ from which there follows}$$

$$\beta = \frac{(y_1 - y_0)\, x_2^2 - (y_2 - y_0)\, x_1^2}{x_1\, x_2^2 - x_2\, x_1^2} \text{ and}$$

$$\gamma = \frac{(y_1 - y_0)\, x_2 - (y_2 - y_0)\, x_1}{x_1^2\, x_2 - x_2^2\, x_1}.$$

We have, consequently,

$$y = y_0 + \left(\frac{(y_1 - y_0)\, x_2^2 - (y_2 - y_0)\, x_1^2}{x_1\, x_2^2 - x_2\, x_1^2}\right) x$$
$$+ \left(\frac{(y_1 - y_0)\, x_2 - (y_2 - y_0)\, x_1}{x_1^2\, x_2 - x_2^2\, x_1}\right) x^2.$$

If the ordinate y_1 lay midway between y_0 and y_2, there would be $x_2 = 2\,x_1$, and then, more simply,

$$y = y_0 - \left(\frac{3\,y_0 - 4\,y_1 + y_2}{2\,x_1}\right) x + \left(\frac{y_0 - 2\,y_1 + y_2}{2\,x_1^2}\right) x^2.$$

If only two pairs of co-ordinates x_0, y_0, and x_1, y_1, be given, we must regard the boundary line $P_0\,P_1$ as a straight line, and consequently, put

$$y = y_0 + \beta x, \text{ as also}$$
$$y_1 = y_0 + \beta x_1,$$

whence there results

$$\beta = \frac{y_1 - y_0}{x_1} \text{ and}$$

$$y = y_0 + \left(\frac{y_1 - y_0}{x_1}\right) x.$$

If it be required to interpolate by construction a fourth ordinate y between the ordinates y_0, y_1, y_2, we must describe a circle through the termini P_0, P_1, P_2, of these ordinates, and take y equal to the ordinate of the same. The centre C of this circle is determined in the following well known manner, viz: let the points P_0, P_1, P_2, be connected together by straight lines, and let perpendiculars be drawn to the centres of these lines; the point C of intersection of these perpendiculars is the centre sought.

If the distances of the point P_1 from the other points P_0 and P_2 be s_0 and s_2, and if the distance $P_1\,K$ of the point P_1 from the con-

necting line $s_1 = P_0 P_2$ be $= h$, there will result for the angle at circumference $\alpha = P_1 P_0 P_2 = \frac{1}{2}$ the angle $P_1 C P_2$ at the centre:

$$sin.\ \alpha = \frac{h}{s_0};$$

and consequently, for the radius of gyration $C P = C P_0 = C P_1 = C P_2$:

$$r = \frac{s_2}{2\ sin.\ \alpha} = \frac{s_0\, s_2}{2 h}.$$

Therefore, we may find the centre C of the circle passing through P_0, P_1, P_2, if, with the radius from P_0, P_1, or P_2, calculated according to this formula, we intersect the perpendicular erected in the middle D of the chord $P_0 P_2$.

ART. 38. The mean of the aggregate ordinates above the line of base $M_0 M_2$, is the height of a rectangle $M_0 M_2 N_2 N_0$ upon the same line of base, which has the same area as the surface $M_0 M_2 P_2 P_1 P_0$, and may, therefore, be easily determined from this area. From Art. 29, we have the same:

$$F = \int_0^{x_2} y \partial x = \int_0^{x_2} (y_0 + \beta x + \gamma x^2)\, \partial x$$

$$= y_0\, x_2 + \frac{\beta x_2^2}{2} + \frac{\gamma x_2^3}{3}$$

$$= y_0\, x_2 + \left(\frac{(y_1 - y_0)\, x_2^2 - (y_2 - y_0)\, x_1^2}{x_1\, x_2^2 - x_2\, x_1^2}\right) \frac{x_2^2}{2}$$

$$+ \left(\frac{(y_1 - y_0)\, x_2 - (y_2 - y_0)\, x_1}{x_1^2\, x_2 - x_2^2\, x_1}\right) \frac{x_2^3}{3}$$

$$= \left(y_0 + \frac{(y_1 - y_0)\, x_2^2}{6\, x_1\, (x_2 - x_1)} - \frac{(y_2 - y_0)\, (3\, x_1 - 2\, x_2)}{6\, (x_2 - x_1)}\right) x_2$$

$$= \left(\frac{y_0 + y_2}{2}\right) x_2 + \left(\frac{(y_1 - y_0)\, x_2 - (y_2 - y_0)\, x_1}{6\, x_1\, (x_2 - x_1)}\right) x_2^2,$$

and consequently, the mean ordinate,

$$y_m = \frac{F}{x_2} = \frac{y_0 + y_2}{2} + \left(\frac{(y_1 - y_0)\, x_2 - (y_2 - y_0)\, x_1}{6\, x_1\, (x_2 - x_1)}\right) x_2.$$

If we had $\frac{y_2 - y_0}{y_1 - y_0} = \frac{x_2}{x_1}$, we should have to consider a rectilinear boundary, and there would then be, simply,

$$F = \left(\frac{y_0 + y_2}{2}\right) x_2,$$

as also,

$$y_m = \frac{y_0 + y_2}{2}.$$

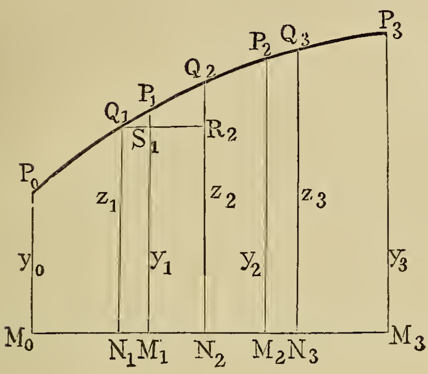
If, further, we had merely $x_2 = 2\, x_1$, therefore, y_1 equally distant from the boundary ordinates y_0 and y_2, there would be

$$F = (y_0 + 4\,y_1 + y_2)\,\frac{x_2}{6} \text{ (vid. Art. 30), and}$$

$$y_m = \frac{y_0 + 4\,y_1 + y_2}{6}.$$

If an area $M_0 M_3 P_3 P_0$, Fig. 50, is determined by four co-ordinates $M_0 P_0 = y_0$, $M_1 P_1 = y_1$, $M_2 P_2 = y_2$, $M_3 P_3 = y_3$, which are at equal distances from each other, the magnitude of the same may be determined *approximately* in the following manner.

Fig. 50.



If we represent the line of base $M_0 M_3$ by x_3, and three ordinates $N_1 Q_1$, $N_2 Q_2$, $N_3 Q_3$, inserted between y_0 and y_3 at equal distances from each other, by z_0, z_1, z_2, we can put the surface approximately:

$$M_0 M_3 P_3 P_0 = F = (\tfrac{1}{2}\,y_0 + z_0 + z_1 + z_2 + \tfrac{1}{2}\,y_3)\,\frac{x_3}{4}.$$

But we have now,

$$\frac{z_1 + z_2 + z_3}{3} = \frac{2\,z_1 + 2\,z_2 + 2\,z_3}{6} = \frac{2\,z_1 + z_2}{6} + \frac{2\,z_3 + z_2}{6} \text{ and}$$

$$y_1 = z_1 + \tfrac{1}{3}\,(z_2 - z_1) = \frac{2\,z_1 + z_2}{3}, \text{ as also } y_2 = \frac{2\,z_3 + z_2}{3};$$

$$\text{hence there follows } \frac{z_0 + z_1 + z_2}{3} = \frac{y_1 + y_2}{2}, \text{ and}$$

$$F = [\tfrac{1}{2}\,y_0 + \tfrac{3}{2}\,(y_1 + y_2) + \tfrac{1}{2}\,y_3]\,\frac{x_3}{4}$$

$$= [y_0 + 3\,(y_1 + y_2) + y_3]\,\frac{x_3}{8}, \text{ as also}$$

$$y_m = \frac{y_0 + 3\,(y_1 + y_2) + y_3}{8}.$$

Whilst the foregoing formula for y_m is applicable when the surface is resolved into an even number of strips, the latter is employed when the *number of these portions is an odd one.*

Consequently, we may also put approximately:

$$\int_c^{c_1} y\,\partial x = \int_c^{c_1} \varphi\,(x)\,\partial x = [y_0 + 3\,(y_1 + y_2) + y_3]\,\frac{c_1 - c}{8},$$

if

$$y_0 = \varphi\,(c),\ y_1 = \varphi\left(\frac{2\,c + c_1}{3}\right),\ y_2 = \varphi\left(\frac{c + 2\,c_1}{3}\right), \text{ and } y_3 = \varphi\,(c_1)$$

are four determined values of the function $y = \varphi\,(x)$.

For example, for $\int_1^2 \frac{\partial x}{x}$ (vid. example, Art. 30), we have $c = 1$, $c_1 = 2$, and $\varphi(x) = \frac{1}{x}$; hence, there follows

$$y_0 = \tfrac{1}{1} = 1,\ y_1 = \frac{3}{2+2} = \tfrac{3}{4},\ y_2 = \frac{3}{1+4} = \tfrac{3}{5}, \text{ and } y_3 = \tfrac{1}{2},$$

and the approximate value of this integral:

$$\int_1^2 \frac{\partial x}{x} = [1 + 3(\tfrac{3}{4} + \tfrac{3}{5}) + \tfrac{1}{2}] \cdot \tfrac{1}{8} = \frac{111}{160} = 0{,}694.$$